Proposed Statistical Model for Scoring and Ranking Sport Tournaments: Racquetball, Squash, and Badminton

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***Abstract*—A class of modification is proposed for calculating a score for each player/team in unbalanced, incomplete paired-comparison sports tournaments. Many papers dealing with balanced incomplete paired-comparison sports tournaments with at most one comparison per pair have appeared since 1950. However, little has been written about unbalanced situations in which the player/the team (object) (*j*) plays unequal number of games against the player/the team (*m*) in a tournament, and the results of all games can be summarized in a Win-Lose matrix $Y=\{Y_{jm}\}$, where $Y_{jm}=1, 0, 1/2$, respectively, according to the player or the team (*j*) wins, losses or draws against the player or the team (*m*). Published papers by Ramanujacharyulu (1964), Cowden (1975), and David (1988) have concentrated on the problem of converting the results of unbalanced incomplete paired-comparison tournaments into rank with little consideration of the main relative ability on each player or team. We suggest (modification) other way of quantifying the outcomes of the games/tournaments, in particular, ratings on a scale, 0–5, 1–10, ect. It is important to consider not only the vector *Vj*(*d*) or the vectors S_j, in scoring and ranking the k teams in such tournaments but also the vector Z_j, where $Z_j=S_j+S_jR_j$, to take into account the ratio of the relative ability of each team (R_j). The proposed modification helps to introduce these methods for use in comparisons/games (tournaments), where the player/team are quantified on a special scale, for example, 0–5 and 1–10. We conclude the following: The scores stabilized to three decimal places at iteration two in Cowden's method $V_j(d)$ Table III. The scores stabilized to three decimal places at iteration two in David's method S_j, and its modification Z_j. The proposed modification (Z_j) has the advantage of removing ties from David's method (S_j), and hence, it is the best method.**

***Keywords*—Paired comparisons, Ranking, Rating on a scale, Scoring, Sport tournaments unbalanced incomplete design.**

I. Introduction

Many papers dealing with balanced incomplete paired-comparison tournaments with at most one comparison per pair have appeared since 1950. Bradley and Terry (1952), Bradley, 1954), and Dykstra (1990) have used Rank analysis of incomplete block of paired comparison. Buhrmann, and Huber (1963) use pairwise comparison and ranking in Tournaments. David and Andrews (1993) introduced a Nonparametric methods of Ranking from paired comparisons. Csato, L. (2013) used Ranking by Pairwise comparisons for Tournaments. However, little has been written about unbalanced situations in which the player/the team (object) (*j*) plays unequal number of games against the player/the team (*m*) in a tournament, and the results of all games are summarized in a Win-Lose matrix $Y=\{Y_{jm}\}$, where $Y_{jm}=1, 0, 1/2$, respectively, according to the player or the team (*j*) wins, losses or draws against the player or the team (*m*). David (1987); (1988) used ranking from unbalanced paired comparison data. Published papers have concentrated on the problem of converting the results of unbalanced, incomplete paired-comparison tournaments into rank with little consideration of the main relative ability on each player or team. Chung and Hwang (1978). Introduce stronger players to win more knock-out tournaments

II. Methodology

Suppose that there are k teams $T_1,\ldots,T_k$ competing in a tournament, and the j_{th} team plays $i=1, 2,\ldots, n$ games. The number of games won by the j_{th} team, in a Win-Lose matrix, is similar to the number of judges who prefer the j_{th} object in the "preference table." Gonzalez Diaz et al. (2013) Described Paired comparisons analysis by axiomatic approach to ranking methods.

Cihan University-Erbil Journal of Humanities and Social Sciences
Volume III, No. 1 (2019), 5 pages
DOI: 10.24086/cuejhss.vol3n1y2019.pp15-19
Received 03 October 2018; Accepted 25 December 2018; Regular research paper: Published 01 June 2019
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A. Generating Scores

Let Y_{jm} be the number of times that team *j* wins against team *m*, and

Let $Y=\{Y_{jm}\}$

If $W_j^{(1)}$ is the row sums of the Win-Lose matrix *Y* (the marginal total number of wins),

Then $W_j^{(1)}=\sum^{k}_{m=1} Y_{jm}=Y.\ 1$

Where 1 is the column vector of *k* ones.

Now let:

$W_j^{(2)}=Y.\ W_j^{(1)}=Y^2.\ 1$

This process may be continued to generate:

$$W_j^{(d)}=Y.\ W_j^{(d-1)}=Y^d.\ 1 \quad (2.1)$$

This approach is based on generating scores by powering the matrix (*Y*) Eq. (2.1). With increasing *d*, rankings based on $W_j^{(d)}$ tend to stabilize. Laslier (1997) introduced tournament solutions and majority voting in paired comparisons.

B. Ramanujacharyulu's Method

Ramanujacharyulu (1964) called $W_j^{(d)}$, the iterated power of order *d* and introduced the corresponding iterated weakness $W_m^{(d)}$, i.e. $W_m^{(1)}$ is the total number of loses for team *j*, which are the column sums of the Win-Lose matrix, such that

$$W_m^{(1)}=\sum^{k}_{j=1} Y_{jm}=Y^t.\ 1 \quad (2.2)$$

Where Y^t is the transpose of the matrix *Y*, as the matrix of losses.

Again $W_m^{(2)}=Y^t.\ W_m^{(1)}=(Y^t)^2.\ 1$

And $W_m^{(d)}=Y^t.\ W_m^{(d-1)}=(Y^t)^{(d)}.\ 1$

C. Cowden's Method

Cowden in 1975, proposed that, instead of multiplying each Y_{jm} by the number of games won by each opponent defeated by him, we may multiply by $V_j^{(d)}$, the weighted proportion of games won, at iteration *d*.

Let $V_j^{(1)}$ be the proportion of games won by the j_{th} team.

Let $V_c^{(1)}$ be the proportion of games lost by the j_{th} team.

So

$V_j^{(1)}=W_j^{(1)}/\{W_j^{(1)}+W_m^{(1)}\}$

And $V_c^{(1)}=W_c^{(1)}/\{W_c^{(1)}+W_j^{(1)}\}$

Let $U_1^{(1)}=Y.\ V_j^{(1)}$

$U_2^{(1)}=Y'.\ V_c^{(1)}$

$V_j^{(2)}=U_1^{(1)}/\{U_1^{(1)}+U_2^{(1)}\}$

$V_c^{(2)}=U_2^{(1)}/\{U_1^{(1)}+U_2^{(1)}\}$

The process may be continued to generate

$$V_j^{(d)}=U_j^{(d-1)}/\{U_j^{(d-1)}+U_c^{(d-1)}\} \quad (2.3)$$

Where $0<V_j^{(d)}<1$, (d=1, 2,…)

And $V_c^{(d)}=1-V_j^{(d)}$

It is convenient (as Cowden said) to use the values of $V_j^{(0)}=V_c^{(0)}=0.5$, in which case $V_j^{(1)}$ is the weighted proportion of games won with iteration (1), and $V_c^{(1)}$ is the weighted proportion of games lost with iteration (1).

The values of $U_j^{(d)}$, $U_c^{(d)}$, $V_j^{(d)}$, and $V_c^{(d)}$, change with each iteration until the values of $V_j^{(d)}$ stabilize. This method of ranking and scoring is not very sensitive to the initial values of $V_j^{(0)}$ and $V_c^{(0)}$.

D. David's Method

David (1987) proposed a simple and useful formula for scoring, and hence ranking, the k_{th} team/player in an incomplete or unbalanced tournament, using the scores vector S_j, such that

$$S_j=W_j^{(2)}-W_m^{(2)}+W_j^{(1)}-W_m^{(1)} \quad (2.4)$$

The idea of this approach is that the vector S_j reflects equally the strength of those teams defeated by T_j and the weakness of those teams by whom T_j was defeated. He considers only (0, 1, 1/2) outcomes.

Ranking according to the score vectors, S_j, Eq. (2.4) has one disadvantage, in that it produces ties among players/teams/objects. Glenn and David (1960) introduced the concept of ties in paired comparison experiments

E. A Proposed Modification to David's Method

We will investigate other ways of quantifying the outcomes of the games/tournaments, in particular, ratings on a scale, 0–5, 0–10, instead of 0, 1 ect.

It is important (The Author suggests) to consider not only the *k*×1 vector $V_j^{(d)}$ (Eq. 2.3) or the *k*×1 vectors S_j (Eq. 2.4), in scoring and ranking *k* teams or contestants in such tournaments but also to take into account the ratio of the relative ability of each team (R_j).

To find this R_j, we work as follows:

Let

K=Number of teams.

H=Number of characters scores (attributes) per team per game

N=Number of games

Then, let

X_{ijmp} i=1, 2,…, n

j, m=1, 2,…, k $j\neq m$

p=1, 2,…, h

Be the score (rating on a scale) of the object j_{th} team playing against the m_{th} team for a characteristic score in the i_{th} game.

Thus

$$M_{ijm}=\sum^{h}_{p=1} X_{ijmp}/\mathrm{h} \quad (2.5)$$

Is the average score per team pergame.

Now, define M_{jm} such that

$M_{jm}=\sum^{h}_{p=1} \sum^{n}_{i=1} X_{ijmp}/hn$

Hence,

$$M_{jm}=\sum^{n}_{i=1} M_{ijm}/n \quad (2.6)$$

Are the means of averages.

The marginal means of the matrix M_{jm} represents the means of the average scores per game for the j_{th} and the m_{th} teams.

i.e.,

$M_j=\sum^{k}_{m=1} M_{jm}/k$ [mean of the average scores for player *j*].

And

$M_m=\sum^{k}_{j=1} M_{jm}/k$ [mean of the average scores for player *j*'s opponents].

Hence, our ratio R_j becomes

$$R_j=M_j/[M_j+M_m] \tag{2.7}$$

As a ratio of the relative ability of team j to team m (as a correction factor).

Cowden (1975) has ranked the players according to the score vector $Vj^{(d)}$, Eq. (2.3). David (1987) has ranked the players to the score vector, S_j Eq. (2.4). However, S_j uses simply the information into the Win-Lose matrix. Now, if the information relating to the actual scores in any game/ tournament is available, we could use Z_j, as a simple combination between S_j and R_j, where:

$$Z_j=S_j*R_j+S_j$$

$$Z_j=S_j*(R_j+1) \tag{2.8}$$

This particular combination of S_j and R_j seems a responsible way for incorporating all the information. It should reduce the number of ties produced by David's method.

III. Application of the Proposed Method (Modification) (Racquetball Tournament)

We can apply the proposed method (modification) in scoring and ranking sport tournament results for a large group of popular sports, such as Racquetball, Squash, and Badminton. All these games involve repeated rallies, with the special rule that points can only be scored by the player who is serving. The game ends when one player earns a fixed number n of points; in Racquetball, Badminton, and Squash, n is often taken to be 21, 15, and 9, respectively, while the other player earns j = 0, 1, 2,..., $n-1$ of points unless players tie at $n-1$ points, when special rules are applied.

In these sports, each game results in a numerical score which should be much more informative than the simple Win-Lose results. These results could be treated as a simple Win-Lose tournament. However, in view of the scoring concept, one could incorporate the Rjextra information to assign a rate to each player, and then select the best.

Strauss and Arnold (1987) have described a numerical method of rating this type of sport. They have developed MLE and moment estimation for rally - winning probabilities for these sports as a simple alternative procedure to fit a model with one parameter Θ_j, (the rate for the j_{th} player), assuming a Bradley-Terry model of the form:

$$P_{jm}=exp\Theta_j/(exp\Theta_j+exp\Theta_m) \tag{3.1}$$

Where P_{jm} is the probability that player j wins a rally over player m. They suggested, for this purpose, that the moment estimator of the expected number of rallies is given by:

$$X^*=X_{\text{j}}+X_m+2SQRT\,(X_jX_m) \tag{3.2}$$

Where X^* is the number of rallies, X_j is the number of points scored by player j, X_m is the number of points scored by player m. We know that n=21, j=0, 1, 2,., $n-1$. Hence, the score of n to j is just like the outcome of X^* Bernoulli trials, where X^* is given by Eq. (3.2) and the number of successes for player (j) and player (m), respectively, are: $n+SQRT\ (n_j)$, $j+SQRT\ (n_j)$. Thus, $X_j+SQRT\ (X_jX_m)$ is the number of successes for player j. $X_m+SQRT\ (X_jX_m)$ is the number of successes for player m. Using Eq. (3.1) and Eq. (3.2), Θ_j (the estimated rate) can be estimated by minimizing the following:

$$\sum_{j=m}^{k}X^*\{log[(X_j+Q)/(X_m+Q)]-(\Theta_j-\Theta_m)\}^2 \tag{3.3}$$

Where

$Q=SQRT\ (X_jX_m)$ the sum being over all the matches in the tournament.

Estimated ratings, Θ_j are displayed in column 1 of Table VI. Our aim is to assign a score to each player, corresponding to his level of ability, such that the probability of victory of one player over the others is a prescribed function of the ratio of points scored by player j to the total points scored by him and by other players with whom we call again the ratio R_j: such that:

$$R_j=X_j/(X_j+X_m) \tag{3.4}$$

Where $Xj=\sum X_{jm}$ is the total points scored by player $jX_m=\sum X_{jm}$ is the total points lost by player j. X_{jm} is the number of points scored by player j against the player m. We use Eq. (2.3), Eq. (2.4), Eq. (2.7), and Eq. (2.8) to find $V_j(d)$, S_j, R_j, and Z_j, respectively, as final scores of the players. These scores are regarded as a descriptive indicator rather than the parameters of an exact model. Now we illustrate this application on data supplied by Strauss and Arnold 1987, as in the following example: Eleven players were taking part in a Racquetball tournament. Each pair of them played g games. A game ends when one player earns a fixed number of points n=21, while his opponent earns J=0, 1, 2, $n-1$ points. Each player's scores are recorded in his row as is shown in Table I:

(Strauss, D. and Arnold, B.C. 1987)

Score "for" a player: Read across the row; score "against:" Read down the column.

Table I has the following aspects:

a. Each game involves two players (paired comparisons)

TABLE I
Racquetball Tournament

Player		1	2	3	4	5	6	7	8	9	10	11	X_j	R_j
1-Currier	1	—	--	--	21	21	--	--	21	21	--	21	84	0.583
2-Strauss	2	—	--	10	12	21	21	--	--	21	21	21	136	0.680
3-Morrison	3	_	21	--	—	13	21	--	21	--	21	10	119	0.546
4-Irving	4	--	11	21	--	8	12	--	20	12	21	4	109	0.408
5-Espinosa	5	17	9	21	21	--	21	21	21	21	21	8	181	0.562
6-Koning	6	--	1	8	21	17	--	21	19	16	14	--	117	0.433
7-Abercrombie	7	--	--	--	--	8	15	--	--	--	21	--	44	0.431
8-Knofflock	8	16	--	15	21	5	21	--	--	15	21	--	114	0.465
9-Mix	9	8	7	--	21	16	21	--	21	--	21	11	126	0.472
10-Davies	10	_	5	3	20	11	21	16	8	14	--	4	102	0.359
11-Carlsson	11	19	10	21	21	21	--	--	--	21	21	--	134	0.629
X_m		60	64	99	158	141	153	58	131	141	182	79		

TABLE II
WIN-LOSE DATA MATRIX

Player	1	2	3	4	5	6	7	8	9	10	11	$W_j^{(1)}$
1	-	--	—	—	1	-	-	1	1	-	1	4
2	--	—	0	1	1	1	-	-	1	1	1	6
3	_	1	—	0	0	1	-	1	-	1	0	4
4	_	0	1	—	0	0	-	0	0	1	0	2
5	0	0	1	1	-	1	1	1	1	1	0	7
6	_	0	0	1	0	-	1	0	0	0	-	2
7	_	—	—	-	0	0	-	-	-	1	-	1
8	0	—	0	1	0	1	-	-	0	1	-	3
9	0	0	-	1	0	1	-	1	-	1	0	4
10	—	0	0	0	0	1	0	0	0	-	0	1
11	0	0	1	1	1	-	-	-	1	1	-	5
W_m (1)	0	1	3	6	3	6	2	4	4	8	2	39

TABLE III
VJ THE SCORES VECTOR (COWDEN'S METHOD)

Player	1	2	3	4	5	6	7	8	9	10	11	$W_j^{(1)}$	$V_j^{(3)}$	Rank
1	-	-	-	-	1	-	-	1	1	-	1	4	1.000	1
2	-	-	0	1	1	1	-	-	1	1	1	6	0.863	4
3	-	1	-	0	0	1	-	1	-	1	0	4	0.586	6
4	-	0	1	-	0	0	-	0	0	1	0	2	0.213	8
5	0	0	1	1	-	1	1	1	1	1	0	7	0.906	3
6	-	0	0	1	0	-	1	0	0	0	-	2	0.100	9
7	-	-	-	-	0	0	-	-	-	1	-	1	0.061	10
8	0	-	0	1	0	1	-	-	0	1	-	3	0.306	7
9	0	0	-	1	0	1	-	1	-	1	0	4	0.685	5
10	-	0	0	0	0	1	0	0	0	-	0	1	0.045	11
11	0	0	1	1	1	-	-	-	1	1	-	5	0.939	2
W_m (1)	0	1	3	6	3	6	2	4	4	8	2			

TABLE IV
S_J THE SCORES VECTOR (DAVID'S METHOD)

Player	1	2	3	4	5	6	7	8	9	10	11	$W_j^{(1)}$	S_j	Rank
1	-	-	-	-	1	-	-	1	1	-	1	4	23	1.5
2	-	-	0	1	1	1	-	-	1	1	1	6	23	1.5
3	-	1	-	0	0	1	-	1	-	1	0	4	2	5.5
4	-	0	1	-	0	0	-	0	0	1	0	2	-19	9
5	0	0	1	1	-	1	1	1	1	1	0	7	18	4
6	-	0	0	1	0	-	1	0	0	0	-	2	-24	10
7	-	-	-	-	0	0	-	-	-	1	-	1	-9	8
8	0	-	0	1	0	1	-	-	0	1	-	3	-6	7
9	0	0	-	1	0	1	-	1	-	1	0	4	2	5.5
10	-	0	0	0	0	1	0	0	0	-	0	1	-30	11
11	0	0	1	1	1	-	-	-	1	1	-	5	20	3
W_m (1)	0	1	3	6	3	6	2	4	4	8	2			

b. Each player does not play against every other one (incomplete tournament).

The results of all games are summarized in a Win-Lose matrix, as shown in Table II:

Now: 1-Use Eq. (3.4) and data in Table III to find R_j, the relative ability of each player. 2-Use Eq. (2.3), and data in Table II to find V_j the scores vector (Cowden's Method) (Table III).

3-Use Eq. (2.4), and data in Table II to find S_j the scores vector (David's Method) (Table IV).

Then, we find Z_j, by Eq. (2.8), the weighted scores vector, we can see the result in Table V.

TABLE V
MODIFIED DAVID'S METHOD

Player	R_j	Z_j	Rank of Z_j
1	0.583	13.41	2
2	0.680	15.64	1
3	0.546	1.09	5
4	0.408	-7.75	9
5	0.562	10.12	4
6	0.433	-10.39	10
7	0.431	-3.88	8
8	0.465	-2.79	7
9	0.472	0.94	6
10	0.359	-10.77	11
11	0.629	12.58	3

TABLE VI
RANKS FOR THREE SCORES VECTORS COMPARED WITH THE RANK OF RATING PARAMETER (Θ_j)

Player	Rating (Θ_i)	$V_j^{(d)}$	S_j	Z_j	Win-game rank ratio %
1-Currier	3	1	1.5	2	100.0
2-Strauss	1	4	1.5	1	85.7
3-Morrison	4	6	5.5	5	57.1
4-Irving	8	8	9	9	25.0
5-Espinosa	5	3	4	4	70.0
6-Koning	9	9	10	10	25.0
7-Abercrombie	10	10	8	8	33.3
8-Knofflock	7	7	7	7	42.8
9-Mix	6	5	5.5	6	50.0
10-Davies	11	11	11	11	11.1
11-Carlsson	2	2	3	3	71.4

Now, rank the results of the three scores vectors V_j, S_j, and Z_j to see if there is any difference between these ranks and Strauss-Arnold's rank (Table VI).

The 6th column of Table VI gives the percentage of games won to a total number of games played. From Table VI, we can see the following results:

1. Player No.1 (Currier) becomes the 2nd, by *Zj* method, instead of 3rd, by Θ_j method.
2. Player No. 2 (Strauss) becomes the 1st, by either method, Z_j and Θ_j•
3. Player No. 3 (Morrison) becomes the 5th, by Z_j method, instead of 4th, by Θ_j method
4. Player No. 11 (Carlsson) becomes the 3rd, by Z_j method, instead of 2nd, by Θ_j method.

These results seem very reasonable because: First: Player no. 11 (Carlsson) has been beaten by player no. 1 (Currier).

Second: The percentages of games won to a total number of games played by player No. 1, and No. 11 are, respectively, 100 and 71.4. A similar story holds for the other players.

Important note: These results give us a very important indicator that the procedure of devising a score for each player by the proposed modification Z_j, Eq. (2.8) is more appropriate than the other methods, since:

a. It incorporates more information about the relative abilities of the players.

b. It uses deferent scale of measurement (0–21), instead (0, 1) data.
c. It removes the ties between players.
d. Strauss and Arnold have developed maximum likelihood and moment estimates for rally-winning probabilities for these types of sport. Such estimates are then used to develop a rating system for the players in a tournament (i.e. modeling, to forecast a rating for the players).

In contrast, we are trying to rate these players using only the actual outcome of a tournament (trying to devise an overall score for each player to see the winner).

IV. Conclusion

The two methods, Cowden's method and David's method, are interested in calculating scores and then ranks for each player from the "preference matrix" Win-Lose matrix, as shown in Table II. The proposed modification helps to introduce these methods for use in comparisons/games (tournaments), where the players are quantified on a special scale, for example, 0–21, instead, (0, 1) data… The scores stabilized to three decimal places at iteration 2 in Cowden's method $V_j^{(d)}$, Table III.

The scores stabilized to three decimal places at iteration 2 in David's method, S_j, and its modification, Z_j (Table IV and Table V).

Table VI shows the ranks for three scores vectors $V_j^{(d)}$, S_j and Z_j at the second iteration. There is a difference between $V_j^{(d)}$ and Z_j, where Player no. 3 becomes the 7th instead of the 6th (Player no. 3 won one game out of five games whereas player no.6 won one game out of four games).

The proposed modification (Z_j) for David's method, (S_j), has the advantage of removing ties from David's method, and it uses a deferent scale of measurement (0–21), instead (0, 1) data. Hence, it is reasonable, good, and the best method.

V. Recommendations

From the results shown in Table VI, we recommend using the proposed method (Z_j) in sport tournaments instead of other methods, because it has many advantages as mentioned before.

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