Kumaraswamy Semicircular Exponentiated Weibull Distribution: Properties and Simulation

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***Abstract*—This paper presents a newly transformed Kumaraswamy semicircular distribution along with their cumulative distribution function, probability density function, and reliability measures based on composing the Kumaraswamy-G family with semicircular exponentiated Weibull distribution. Furthermore, the most important statistical properties, including moments, characteristic function, trigonometric moments, quantile function, simulated data, reliability stress strength model, Shannon and relative entropies, are also obtained. A simulation study is conducted to empirically detect the performances of the maximum likelihood estimates of the parameters within different default values and samples size. The results give grounds for optimism about the distribution's stability and flexibility in practical applications. In addition, the new distribution is an extension of the semicircular exponentiated Weibull distribution.**

***Keywords*—Circular distributions, Kumaraswamy-G family, Semicircular exponentiated Weibull, Statistical properties, Stress strength, Entropies.**

I. INTRODUCTION

Circular "angular or directional" data have been collected from a range of fields that measure angle or direction, such as zoology, geology, biology, earth science, and so on. Many detailed circular models are contained in [1], such instances are bird navigation, differences at the beginning of leukemia, texture orientation data, and wind directions. However, angular data may not necessitate whole circular models for modeling in some cases [2][3]. Random variables with values on a semicircle, for example, are sufficient to convey angular data when sea turtles emerge from the ocean in search of a dry land nesting spot, or when an airplane is lost but its departure and arrival headings are recognized.

Several circular and semicircular distributions respectively with x∈(-∞,∞) and x∈(0,∞) are generated using wrapping, conditioning/characterizing, offsetting, or inverse stereographic projection methods. The symmetric circular [4], circular extreme value [5], modified wrapped exponential [6], logistic circular [7], stereographic semicircular Gamma [8], logistic semicircular [9], semicircular extreme-value [10], semicircular arc tan [11], and semicircular exponentiated Weibull [12] are few examples of the circular and semicircular distributions. The aim of this paper is to compose the Kumaraswamy-G family with semicircular exponentiated Weibull distribution to present a new semicircular distribution projected onto [0,π) that can be used to study semicircular data.

II. KUMARASWAMY SEMICIRCULAR EXPONENTIATED WEIBULL DISTRIBUTION

Cordeiro and de Castro [13] introduced Kumaraswamy G-family as a new family of generalized distributions that improve the flexibility of well-known baseline distributions. The cdf and pdf related to the Kumaraswamy G-family are given respectively as:

$$F(x)_K = 1-\left(1-G^a(x)\right)^b;\ 0<x<1; a,b>0 \qquad (1)$$

$$f(x)_K = abg(x)G^{a-1}(x)\left(1-G^a(x)\right)^{b-1} \qquad (2)$$

where G(x) and g(x) represent the cdf and pdf of the baseline distribution.

Recently, Oleiwi et al. [12] used inverse stereographic projection on three parameters (λ, α, γ) exponentiated Weibull distribution with support on R+ and x=v tan (θ/2) to present a new semicircular distribution projected onto [0,π) that can be used to study semicircular data, called Semicircular Exponentiated Weibull (SEW). The formulas of the pdf and cdf of the SEW distribution with three parameters λ, α, and β=γ/v with radius v∈R+ are given respectively by:

$$F(\theta)_{SEW} = \left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha};\ \theta\in[0,\pi),\lambda,\alpha,\beta>0 \qquad (3)$$

$$f(\theta)_{SEW} = \frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\sec^2\left(\frac{\theta}{2}\right) \qquad (4)$$

By composing the Kumaraswamy-G family with semicircular exponentiated Weibull distribution, a new semicircular distribution called Kumaraswamy semicircular exponentiated Weibull (KSEW) mapped onto [0, π) can be introduced.

A. *The cdf and pdf*

Consider the G(x) and g(x) in (1) and (2) as the cdf and pdf of the SEW given in (3) and (4) [12], then the cdf and pdf of the KSEW distribution with five parameters a,b,λ ,α, and β=γ/v are given by:

$$F(\theta)_{KSEW} = 1-\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b};\theta\in[0,\pi),a,b,\lambda,\alpha,\beta>0 \quad (5)$$

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$$f(\theta)_{KSEW} = \frac{ab\lambda\alpha}{2\beta^{\lambda}} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}} \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} \sec^{2}\left(\frac{\theta}{2}\right) \quad (6)$$

B. *Reliability Measures*

Based on (5) and (6), the KSEW reliability measures "reliability function, hazard function, cumulative hazard function, and reverse hazard function" can easily be found respectively as (see [14] and [15]).

$$\tau_1(\theta)_{KSEW} = 1 - F(\theta)_{KSEW} = \left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b} \quad (7)$$

$$\tau_2(\theta)_{KSEW} = \frac{f(\theta)_{KSEW}}{1 - F(\theta)_{KSEW}} = \frac{ab\lambda\alpha}{2\beta^{\lambda}} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}$$
$$\left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1} \frac{\sec^{2}\left(\frac{\theta}{2}\right)}{1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}} \quad (8)$$

$$\tau_3(\theta)_{KSEW} = -\ln(1 - F(\theta)_{KSEW})$$
$$= -b\ln\left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right) \quad (9)$$

$$\tau_4(\theta)_{KSEW} = \frac{f(\theta)_{KSEW}}{F(\theta)_{KSEW}} = \frac{ab\lambda\alpha}{2\beta^{\lambda}} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}} \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} \frac{\sec^{2}\left(\frac{\theta}{2}\right)}{1 - \left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b}} \quad (10)$$

C. *The r^{th} Moment*

The r^{th} non-central moment of the KSEW distribution, $E(\theta^{r})_{KSEW}$, can be obtained through the pdf in (6) with $\sec^{2}(\theta/2)=1+\tan^{2}(\theta/2)$ as follows:

$$E(\theta^{r})_{KSEW} = \int_{0}^{\pi} \theta^{r} f(\theta)_{KSEW}\, d\theta$$
$$= \frac{ab\lambda\alpha}{2\beta^{\lambda}} \int_{0}^{\pi} \theta^{r} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}} \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1 - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} \left(1 + \tan^{2}\left(\frac{\theta}{2}\right)\right) d\theta \quad (11)$$

Using the transformation $x = v\tan\left(\frac{\theta}{2}\right)$, $\theta = 2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta = \frac{2}{v+\frac{x^{2}}{v}} dx$, then (11) will be:

$$E(\theta^{r})_{KSEW} = 2^{r} \frac{ab\lambda\alpha}{(\beta v)^{\lambda}} \int_{0}^{\infty} \left(\tan^{-1}\left(\frac{x}{v}\right)\right)^{r} x^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1 - \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} dx$$

Based on expanded formula:

$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\,(k!)^{2}\,(2k+1)} \left(\frac{x^{2}}{x^{2}+1}\right)^{k+\frac{1}{2}}$; $x^{2} < \infty$, the $E(\theta^{r})_{KSEW}$ will be:

$$E(\theta^{r})_{KSEW} = 2^{r} \frac{ab\lambda\alpha}{(\beta v)^{\lambda}} \int_{0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^{2}(2k+1)} \left(\frac{\left(\frac{x}{v}\right)^{2}}{\left(\frac{x}{v}\right)^{2}+1}\right)^{k+\frac{1}{2}}\right)^{r}$$
$$x^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha-1} \left(1 - \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} dx$$

Let $u = \frac{\left(\frac{x}{v}\right)^{2}}{\left(\frac{x}{v}\right)^{2}+1} \Rightarrow x = v\left(\frac{1}{u} - 1\right)^{-1/2} \Rightarrow dx = \frac{v}{2u^{2}}\left(\frac{1}{u} - 1\right)^{-3/2} du$

then

$$E(\theta^{r})_{KSEW} = 2^{r-1} \frac{ab\lambda\alpha}{\beta^{\lambda}} \int_{0}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^{2}(2k+1)} u^{k}\right)^{r} u^{\frac{r}{2}-2} \left(\frac{1}{u} - 1\right)^{-\frac{\lambda}{2}-1} e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}$$
$$\left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{a\alpha-1} \left(1 - \left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{a\alpha}\right)^{b-1} du$$

Based on expansion formula, $(1-z)^{n} = \sum_{k=0}^{\infty}(-1)^{k}\binom{n}{k} z^{k}$; $|z| < 1, n > 0$, and the Binomial coefficients $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, we get

$$\left(1 - \left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{\frac{-\lambda}{2}}}\right]^{a\alpha}\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{\frac{-\lambda}{2}}}\right]^{ia\alpha}.$$

Now

$$E(\theta^{r})_{KSEW}$$
$$= 2^{r-1} \frac{ab\lambda\alpha}{\beta^{\lambda}} \sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i} \int_{0}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^{2}(2k+1)} u^{k}\right)^{r}$$
$$u^{\frac{r}{2}-2}\left(\frac{1}{u} - 1\right)^{-\frac{\lambda}{2}-1} e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}} \left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{(i+1)a\alpha-1} du$$

Similarly, $\left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{\frac{-\lambda}{2}}}\right]^{(i+1)a\alpha-1} = \sum_{j=0}^{\infty}(-1)^{j}\binom{(i+1)a\alpha-1}{j} e^{-\frac{j}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{\frac{-\lambda}{2}}}$

Then

$$E(\theta^{r})_{KSEW} = 2^{r-1} \frac{ab\lambda\alpha}{\beta^{\lambda}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{(i+1)a\alpha-1}{j}\binom{b-1}{i}$$
$$\int_{0}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^{2}(2k+1)} u^{k}\right)^{r} u^{\frac{r}{2}-2}\left(\frac{1}{u} - 1\right)^{-\frac{\lambda}{2}-1} e^{-\frac{(j+1)}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}}} du$$

Furthermore, based on expanded exponential function, $E(\theta^{r})_{KSEW}$ will be

$$E(\theta^{r})_{KSEW} = 2^{r-1} \frac{ab\lambda\alpha}{\beta^{\lambda}} \sum_{i,j,m=0}^{\infty} \frac{(-1)^{i+j+m}(j+1)^{m}}{m!\;\beta^{m\lambda}} \binom{(i+1)a\alpha-1}{j}\binom{b-1}{i}$$
$$\int_{0}^{1} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^{2}(2k+1)} u^{k}\right)^{r} u^{\frac{r}{2}-2}\left(\frac{1}{u} - 1\right)^{-(m+1)\frac{\lambda}{2}-1} du$$

Using Newton Binomial formula, then

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$$\left(\frac{1}{u}-1\right)^{-(m+1)\frac{\lambda}{2}-1}=\sum_{\ell=0}^{\infty}\binom{-(m+1)\frac{\lambda}{2}-1}{\ell}(-1)^{\ell}\left(\frac{1}{u}\right)^{-(m+1)\frac{\lambda}{2}-\ell-1}$$

Now, $E(\theta^r)_{KSEW}$ will be

$$E(\theta^r)_{KSEW}=2^{r-1}\frac{ab\lambda\alpha}{\beta^{\lambda}}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+m+\ell}(j+1)^m}{m!\ \beta^{m\lambda}}\binom{(i+1)a\alpha-1}{j}$$
$$\binom{b-1}{i}\binom{-(m+1)\frac{\lambda}{2}-1}{\ell}\int_0^1\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{1}{2}(r+(m+1)\lambda)+\ell-1}du$$

According to circumstance, $(\sum_{k=0}^{\infty}a_k u^k)^r=\sum_{k=0}^{\infty}b_k u^k; r$ is a natural number, $b_0=a_0^r$ and $b_s=\frac{1}{a_0 s}\sum_{k=1}^{s}(kr-s+k)\,a_k b_{s-k}, s\geq 1$, the $E(\theta^r)_{KSEW}$ with $a_k=\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}$ will be

$$E(\theta^r)_{KSEW}=2^{r-1}ab\lambda\alpha\sum_{i,j,m,\ell,k=0}^{\infty}\frac{(-1)^{i+j+m+\ell}(j+1)^m b_k}{m!\ \beta^{\lambda(m+1)}\left(k+\ell+\frac{1}{2}(r+(m+1)\lambda)\right)}$$
$$\binom{(i+1)a\alpha-1}{j}\binom{b-1}{i}\binom{-(m+1)\frac{\lambda}{2}-1}{\ell}\qquad(12)$$

D. The Characteristic Function

The characteristic function of the $KSEW$ distribution can be obtained as:

$$\varphi_p(\theta)_{KSEW}=E\left(e^{ip\theta}\right)_{KSEW}=\int_0^{\pi}e^{ip\theta}\,f(\theta)_{KSEW}\,d\theta\qquad(13)$$

Also, the $\varphi_p(\theta)_{SKEW}$ in (11) can be rewritten (see [14] and [15]) as:

$\varphi_p(\theta)_{KSEW}=\int_0^{\pi}\sum_{r=0}^{\infty}\frac{(ip\theta)^r}{r!}f(\theta)_{SKEW}\,d\theta=\sum_{r=0}^{\infty}\frac{(ip)^r}{r!}E(\theta^r)_{KSEW}.$

Based on (12), the $\varphi_p(\theta)_{KSEW}$ is given by

$$\varphi_p(\theta)_{KSEW}=ab\lambda\alpha\sum_{i,j,m,\ell,k,r=0}^{\infty}\frac{(-1)^{i+j+m+\ell}(ip)^r 2^{r-1}(j+1)^m b_k}{m!\,r!\,\beta^{\lambda(m+1)}\left(k+\ell+\frac{1}{2}(r+(m+1)\lambda)\right)}$$
$$\binom{(i+1)a\alpha-1}{j}\binom{b-1}{i}\binom{-(m+1)\frac{\lambda}{2}-1}{\ell}\qquad(14)$$

Furthermore, the p^{th} ; $p=0,\pm1,\pm2,\ldots$ non-central trigonometric moments can be introduced from

$$\varphi_p(\theta)_{KSEW}=\nabla_p+i\,\Delta_p=E\left(\cos(p\theta)\right)_{KSEW}+i\,E\left(\sin(p\theta)\right)_{KSEW}$$

with

$$\Delta_p=E(\sin(p\theta))_{KSEW}=\sum_{l=0}^{\infty}\frac{(-1)^l p^{2l+1}}{(2l+1)!}E(\theta^{2l+1})_{KSEW}$$

$$\nabla_p=E(\cos(p\theta))_{KSEW}=\sum_{l=0}^{\infty}\frac{(-1)^l\,p^{2l}}{(2l)!}E(\theta^{2l})_{KSEW}$$

where $E(\theta^{2l+1})_{KSEW}$ and $E(\theta^{2l})_{KSEW}$ as in (12) respectively with $r=2l+1$ and $r=2l$.

E. Quantile Function and Simulated Data

Through inverting the cdf in (5), the quantile function of the $KSEW$ distribution can be attained as follows

$$\theta_{(q)-KSEW}=\mathrm{Q}(q)=2\tan^{-1}\left(\beta\left[-\ln\left(1-\left[1-(1-u)^{1/b}\right]^{1/a\alpha}\right)\right]^{1/\lambda}\right)\qquad(15)$$

The median of the $KSEW$ random variable can be gained from (15) by setting $q=1/2$ as

$$Median_{KSEW}=\mathrm{Q}(1/2)=2\tan^{-1}\left(\beta\left[-\ln\left(1-\left[1-(1/2)^{1/b}\right]^{1/a\alpha}\right)\right]^{1/\lambda}\right)\qquad(16)$$

By replacing q with u, a random variable that follows SEW distribution can be simulated as

$$\theta_{KSEW}=2\tan^{-1}\left(\beta\left[-\ln\left(1-\left[1-(1-u)^{1/b}\right]^{1/a\alpha}\right)\right]^{1/\lambda}\right)\qquad(17)$$

where u is an interval-based uniform random number $(0,1)$.

As related measures of quantile function with specified values, the Galton's coefficient of skewness [16] and Moors's coefficient of kurtosis [17], can be easily computed.

F. Stress Strength Model

Consider two independent random variables, Y: $stress$ and Z: strength, that follow $KSEW$ distribution with different parameters. The reliability stress strength model of the $KSEW$ distribution can be obtained (see [14] and [15]) by

$$SS_{KSEW}=P(Y<Z)_{KSEW}=E(F_Y(\theta)_{KSEW})=\int_0^{\pi}F_Y(\theta)_{KSEW}f_Z(\theta)_{KSEW}\ d\theta\qquad(18)$$

where $F_Y(\theta)_{KSEW}$ represents the cdf of the $KSEW$ distribution as in (5) with parameters $a_1,b_1,\lambda_1,\alpha_1,\beta_1$, i.e. $F_Y(\theta)_{KSEW}=1-\left(1-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{a_1\alpha_1}\right)^{b_1}$ and $f_Z(\theta)_{KSEW}$ represents the pdf of the $KSEW$ distribution with parameters a,b,λ,α,β as in (6). Now

$$\left(1-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{a_1\alpha_1}\right)^{b_1}=\sum_{r=0}^{\infty}(-1)^r\binom{b_1}{r}\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{ra_1\alpha_1}$$
$$=\sum_{r=0}^{\infty}(-1)^r\binom{b_1}{r}\sum_{m=0}^{\infty}(-1)^m\binom{ra_1\alpha_1}{m}e^{-m\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}$$
$$=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{r+m+z}}{z!}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^z\tan^{z\lambda_1}\left(\frac{\theta}{2}\right)$$

and $F_Y(\theta)_{KSEW}$ will be

$$F_Y(\theta)_{KSEW}=1-\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{r+m+z}}{z!}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^z\tan^{z\lambda_1}\left(\frac{\theta}{2}\right)\qquad(19)$$

Substituting (19) in (18), yields

$$SS_{KSEW}=1-\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{r+m+z}}{z!}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^z\int_0^{\pi}\tan^{z\lambda_1}\left(\frac{\theta}{2}\right)f(\theta)_{KSEW}\,d\theta\qquad(20)$$

Inserting (6) with $\sec^2\left(\frac{\theta}{2}\right)=1+\tan^2\left(\frac{\theta}{2}\right)$ in (20), SS_{KSEW} will be

$$SS_{KSEW}=1-\frac{ab\lambda\alpha}{2\beta^{\lambda}}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{r+m+z}}{z!}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^z$$
$$\int_0^{\pi}\tan^{z\lambda_1+\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)d\theta$$

Using the transformation, $x=v\tan\left(\frac{\theta}{2}\right)$, then

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$$SS_{KSEW} = 1 - \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{r+m+z}}{z!}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^{z}\int_0^{\infty}\left(\frac{x}{v}\right)^{z\lambda_1}$$
$$\frac{ab\lambda\alpha}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha-1}\left(1-\left[1-e^{-\left(\frac{x}{v\beta}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} dx$$

Since $\left(1-\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{ia\alpha}$, now

$$SS_{KSEW} = 1 - \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{r+m+z+i}}{z!\,(i+1)}\binom{b-1}{i}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^{z}$$
$$b\int_0^{\infty}\left(\frac{x}{v}\right)^{z\lambda_1}\frac{(i+1)a\alpha\lambda}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{(i+1)a\alpha-1} dx$$

But $\frac{(i+1)a\alpha\lambda}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{(i+1)a\alpha-1}$ represents the pdf of EW distribution (see [18] and [19]) with parameters $\lambda, (i+1)a\alpha$, and $\gamma = \beta v$. Then, $\int_0^{\infty}\left(\frac{x}{v}\right)^{z\lambda_1}\frac{(i+1)a\alpha\lambda}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{(i+1)a\alpha-1} dx$ will equal to

$$A = \begin{cases} (i+1)a\alpha\beta^{z\lambda_1}\Gamma\left(\frac{z\lambda_1}{\lambda}+1\right)\sum_{j=0}^{(i+1)a\alpha-1}\binom{(i+1)a\alpha-1}{j}(-1)^j(j+1)^{-\frac{z\lambda_1}{\lambda}-1}; (i+1)a\alpha \in N \\ (i+1)a\alpha\beta^{z\lambda_1}\,\Gamma\left(\frac{z\lambda_1}{\lambda}+1\right)\sum_{j=0}^{\infty}\frac{{}_{(i+1)a\alpha-1}P_j}{j!}(-1)^j\,(j+1)^{-\frac{z\lambda_1}{\lambda}-1} \qquad ; (i+1)a\alpha \notin N \end{cases} \tag{21}$$

where ${}_{(i+1)a\alpha-1}P = \big((i+1)a\alpha-1\big)\big((i+1)a\alpha-2\big)\ldots\big((i+1)a\alpha-j\big)$ with N is the natural numbers.

Therefore, the reliability stress strength model of the $TSEW$ is given by

$$SS_{KSEW} = 1 - b\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{z=0}^{\infty}\sum_{i=0}^{\infty}A\,\frac{(-1)^{r+m+z+i}}{z!\,(i+1)}\binom{b-1}{i}\binom{b_1}{r}\binom{ra_1\alpha_1}{m}\left(\frac{m}{\beta_1^{\lambda_1}}\right)^{z} \tag{22}$$

where A as in (21).

G. *Shannon and Relative Entropies*

The Shannon entropy SH of the $KSEW$ distribution can be obtained through the following formula:

$$SH_{KSEW} = E\left(\ln\left(\frac{1}{f(\theta)_{KSEW}}\right)\right) = E\big(-\ln(f(\theta)_{KSEW})\big) \tag{23}$$

Recall $f(\theta)_{KSEW}$ in (6) and take the natural logarithm, yields

$$\ln(f(\theta)_{KSEW}) = \ln\left(\frac{ab\lambda\alpha}{2\beta^{\lambda}}\right) + (\lambda-1)\ln\left(\tan\left(\frac{\theta}{2}\right)\right) - \left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}$$
$$+(a\alpha-1)\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right) + (b-1)\ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right) + 2\ln\left(\sec\left(\frac{\theta}{2}\right)\right)$$

The Shannon entropy in (23) will be

$$SH_{KSEW} = \ln\left(\frac{2\beta^{\lambda}}{ab\alpha\lambda}\right) - (\lambda-1)E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right) + \frac{1}{\beta^{\lambda}}E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right)$$
$$-(a\alpha-1)E\left(\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)\right)$$
$$-(b-1)E\left(\ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)\right) - 2E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right) \tag{24}$$

Let $\dot{I}_1 = E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right), \dot{I}_2 = E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right), \dot{I}_3 = E\left(\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)\right)$,
$\dot{I}_4 = E\ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)$ and $\dot{I}_5 = E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right)$.

Now, for $\dot{I}_1 = \int_0^{\pi}\ln\left(\tan\left(\frac{\theta}{2}\right)\right)f(\theta)_{KSEW}\,d\theta$, recall (4) with $\sec^2\left(\frac{\theta}{2}\right) = 1+\tan^2\left(\frac{\theta}{2}\right)$, then

$$\dot{I}_1 = \frac{ab\alpha\lambda}{2\beta^{\lambda}}\int_0^{\pi}\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)d\theta$$

Using the transformation, $x = v\tan\left(\frac{\theta}{2}\right)$, then

$$\dot{I}_1 = \frac{ab\alpha\lambda}{v^{\lambda}\beta^{\lambda}}\int_0^{\infty}(\ln(x)-\ln(v))\,x^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha-1}\left(1-\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} dx$$

Using $\left(1-\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left(1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right)^{ia\alpha}$, then $\dot{I}_1$ will be

$$\dot{I}_1 = \frac{ab\alpha\lambda}{v^{\lambda}\beta^{\lambda}}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\int_0^{\infty}(\ln(x)-\ln(v))\,x^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left(1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right)^{(i+1)a\alpha-1} dx$$

Again, $\left(1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right)^{(i+1)a\alpha-1} = \sum_{j=0}^{\infty}(-1)^j\binom{(i+1)a\alpha-1}{j}e^{-j\left(\frac{x}{\beta v}\right)^{\lambda}}$. Now

$$\dot{I}_1 = \frac{ab\lambda\alpha}{v^{\lambda}\beta^{\lambda}}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}\int_0^{\infty}(\ln(x)-\ln(v))\,x^{\lambda-1}e^{-(j+1)\left(\frac{x}{\beta v}\right)^{\lambda}} dx$$

Let $u = \left(\frac{x}{\beta v}\right)^{\lambda} \Rightarrow x = \beta v u^{1/\lambda} \Rightarrow dx = \frac{\beta v}{\lambda}u^{\frac{1}{\lambda}-1}du$, then

$$\dot{I}_1 = ab\alpha\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}\left[\frac{\ln(\beta)}{j+1}+\frac{1}{\lambda}\int_0^{\infty}\ln(u)\,e^{-(j+1)u}du\right]$$

Since $\int_0^{\infty}x^{s-1}\ln(x)\;e^{-mx}\,dx = m^{-s}\,\Gamma(s)(\psi(s)-\ln(m))$ where $\Gamma(s)$ and $\psi(s)$ represent the gamma and digamma functions, then $\int_0^{\infty}\ln u\,e^{-(j+1)u}\;du$ can be rewritten as $\int_0^{\infty}\ln(u)\,e^{-(j+1)u}du = \frac{1}{j+1}(\psi(1)-\ln(j+1))$, then

$$\dot{I}_1 = ab\alpha\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{j+1}\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}\left[\ln(\beta)+\frac{1}{\lambda}(\psi(1)-\ln(j+1))\right] \tag{25}$$

For $\dot{I}_2 = \int_0^{\pi}\tan^{\lambda}\left(\frac{\theta}{2}\right)f(\theta)_{KSEW}d\theta$, recall (6) with $\sec^2\left(\frac{\theta}{2}\right) = 1+\tan^2\left(\frac{\theta}{2}\right)$, then

$$\dot{I}_2 = \int_0^{\pi}\tan^{\lambda}\left(\frac{\theta}{2}\right)\frac{ab\lambda\alpha}{2\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha-1}$$
$$\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)d\theta$$

Using the transformation, $x = v\tan\left(\frac{\theta}{2}\right)$, then

$$\dot{I}_2 = \int_0^{\infty}x^{\lambda}\frac{ab\lambda\alpha}{\beta^{\lambda}v^{\lambda+1}}\left(\frac{x}{v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha-1}\left(1-\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} dx$$

Since $\left(1-\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{a\alpha}\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left(1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right)^{ia\alpha}$, then

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$$\dot{I}_2 = \frac{b}{v^\lambda}\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}\binom{b-1}{i}\int_0^\infty x^\lambda \frac{(i+1)a\alpha\lambda}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^\lambda}\left[1-e^{-\left(\frac{x}{\beta v}\right)^\lambda}\right]^{(i+1)a\alpha-1} dx$$

But $\frac{(i+1)a\alpha\lambda}{\beta v}\left(\frac{x}{\beta v}\right)^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^\lambda}\left[1-e^{-\left(\frac{x}{\beta v}\right)^\lambda}\right]^{(i+1)a\alpha-1}$ represents the pdf of EW distribution with parameters $\lambda, (i+1)a\alpha$, and $\gamma = \beta v$. Therefore, $\dot{I}_2 = \frac{b}{v^\lambda}\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}\binom{b-1}{i} E\left(X^\lambda\right)_{EW}$ (see [18] and[19]) and then,

$$\dot{I}_2 = \begin{cases} ab\alpha\beta^\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{(i+1)a\alpha-1}\frac{(-1)^{i+j}}{(j+1)^2}\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}; \ (i+1)a\alpha \in N \\ ab\alpha\beta^\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!(j+1)^2}\binom{b-1}{i}{}_{(i+1)a\alpha-1}P_j \quad ; \ (i+1)a\alpha \notin N \end{cases} \tag{26}$$

For $\dot{I}_3 = \int_0^\pi \ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right) f(\theta)_{KSEW} d\theta$, based on $\ln(1-z) = -\sum_{k=1}^{\infty}\frac{1}{k} z^k ; \ |z| < 1$
and expanded exponential function, we get

$$\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right) = -\sum_{h=1}^{\infty}\frac{1}{h} e^{-h\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda} = \sum_{h=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}h^{k-1}}{k!\,\beta^{k\lambda}}\tan^{k\lambda}\left(\frac{\theta}{2}\right)$$

Now, $\dot{I}_3 = \sum_{h=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}h^{k-1}}{k!\beta^{k\lambda}} E\left(\tan^{k\lambda}\left(\frac{\theta}{2}\right)\right)_{KSEW}$, where $E\left(\tan^{k\lambda}\left(\frac{\theta}{2}\right)\right)$ can be obtained similarly to $\dot{I}_2$ with $\lambda = k\lambda$, i.e.

$$E\left(\tan^{k\lambda}\left(\frac{\theta}{2}\right)\right) = \begin{cases} ab\alpha\beta^{k\lambda}\Gamma(k+1) \sum_{i=0}^{\infty}\sum_{j=0}^{(i+1)a\alpha-1}\frac{(-1)^{i+j}}{(j+1)^{k+1}}\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}; \ (i+1)a\alpha \in N \\ ab\alpha\beta^{k\lambda}\Gamma(k+1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!(j+1)^{k+1}}\binom{b-1}{i}{}_{(i+1)a\alpha-1}P_j \ ; \ (i+1)a\alpha \notin N \end{cases} \tag{27}$$

Therefore,

$$\dot{I}_3 = \begin{cases} ab\alpha \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty}\sum_{j=0}^{(i+1)a\alpha-1}\frac{(-1)^{i+j+k+1}h^{k-1}}{k!(j+1)^{k+1}}\Gamma(k+1)\binom{b-1}{i}\binom{(i+1)a\alpha-1}{j}; (i+1)a\alpha \in N \\ ab\alpha \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k+1}h^{k-1}}{k!j!(j+1)^{k+1}}\Gamma(k+1)\binom{b-1}{i}{}_{(i+1)a\alpha-1}P_j \ ; \ (i+1)a\alpha \notin N \end{cases} \tag{28}$$

For $\dot{I}_4 = \int_0^\pi \ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right]^{a\alpha}\right) f(\theta)_{KSEW}\, d\theta$ using the expanded formula $\ln(\cos(z)) = -\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)}{k(2k)!}|B_{2k}| z^{2k} \ ; \ z^2 < \frac{\pi^2}{4}$ where $B_{2k} \cong (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}$ the Bernoulli numbers as and the expanded formula of exponential function, we get

$$\ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right]^{a\alpha}\right) = \sum_{r,k=0}^{\infty}\sum_{z=1}^{\infty}\frac{(-1)^{r+k+1}}{k!\,z}\left(\frac{r}{\beta^\lambda}\right)^k\binom{za\alpha}{r}\tan^{k\lambda}\left(\frac{\theta}{2}\right)$$

Now $\dot{I}_4 = \sum_{r,k=0}^{\infty}\sum_{z=1}^{\infty}\frac{(-1)^{r+k+1}}{k!z}\left(\frac{r}{\beta^\lambda}\right)^k\binom{za\alpha}{r} E\left(\tan^{k\lambda}\left(\frac{\theta}{2}\right)\right)$, where $E\left(\tan^{k\lambda}\left(\frac{\theta}{2}\right)\right)$ as in (27). Therefore,

$$\dot{I}_4 = \begin{cases} ab\alpha \sum_{i,r,k=0}^{\infty}\sum_{j=0}^{(i+1)a\alpha-1}\sum_{z=1}^{\infty}\frac{(-1)^{i+j+r+k+1}}{k!(j+1)^{k+1}z} r^k\Gamma(k+1)\binom{za\alpha}{r}\binom{b-1}{i} \\ \qquad \binom{(i+1)a\alpha-1}{j}; \ (i+1)a\alpha \in N \\ ab\alpha \sum_{i,r,k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{z=1}^{\infty}\frac{(-1)^{i+j+r+k+1}}{k!j!(j+1)^{k+1}z} r^k\Gamma(k+1)\binom{za\alpha}{r}\binom{b-1}{i} \\ \qquad {}_{(i+1)a\alpha-1}P_j \ ; \ (i+1)a\alpha \notin N \end{cases}$$

Finally for $\dot{I}_5 = \int_0^\pi \ln\left(\sec\left(\frac{\theta}{2}\right)\right) f(\theta)_{SEW}\, d\theta = \int_0^\pi -\ln\left(\cos\left(\frac{\theta}{2}\right)\right) f(\theta)_{SEW}\, d\theta$ and as before $-\ln\left(\cos\left(\frac{\theta}{2}\right)\right) = \sum_{t=1}^{\infty}\frac{2^{2t-1}(2^{2t}-1)}{t(2t)!}|B_{2t}|\left(\frac{\theta}{2}\right)^{2t} ; \ |\theta| < \pi$, and B_{2t} represent the Bernoulli numbers. Now $\dot{I}_5 = \int_0^\pi \ln\left(\sec\left(\frac{\theta}{2}\right)\right) f(\theta)_{KSEW} d\theta = \sum_{t=1}^{\infty}\frac{2^{2t-1}}{2t(2t)!}|B_{2t}|\, E(\theta^{2t})_{KSEW}$ where $E(\theta^{2t})$ as in (12) with $r = 2t$, then

$$\dot{I}_5 = ab\lambda\alpha \sum_{i,j,m,\ell,k=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{i+j+m+\ell}(j+1)^m 2^{2(t-1)}(2^{2t}-1)b_k|B_{2t}|}{t(2t)!\,m!\,\beta^{\lambda(m+1)}\left(k+\ell+\frac{1}{2}(2t+(m+1)\lambda)\right)} \binom{(i+1)a\alpha-1}{j}\binom{b-1}{i}\binom{-(m+1)\frac{\lambda}{2}-1}{\ell} \tag{30}$$

Therefore, the Shannon entropy of the $KSEW$ distribution in (24) can be obtained as

$$SH_{KSEW} = \ln\left(\frac{2\beta^\lambda}{ab\alpha\lambda}\right) - (\lambda-1)\dot{I}_1 + \frac{1}{\beta^\lambda}\dot{I}_2 - (a\alpha-1)\dot{I}_3 - (b-1)\dot{I}_4 - 2\dot{I}_5 \tag{31}$$

where $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$, $\dot{I}_4$ and $\dot{I}_5$ are respectively given in $(25), (26), (28), (29)$ and (30).

The relative entropy of the $KSEW$ distribution can be obtained by

$$RE_{KSEW} = E\left(\ln\left[\frac{f(\theta)_{KSEW}}{f_1(\theta)_{KSEW}}\right]\right) = \int_0^\pi \ln\left[\frac{f(\theta)_{KSEW}}{f_1(\theta)_{KSEW}}\right] f(\theta)_{KSEW} d\theta \tag{32}$$

Taking the natural logarithm of the $f(\theta)_{KSEW}$ in (6) with five parameters $(a, b, \lambda, \alpha, \beta)$ relative to the $f_1(\theta)_{KSEW}$ with five parameters $(a_1, b_1, \lambda_1, \alpha_1, \beta_1)$, then the RE of the $KSEW$ distribution in (32) can be obtained as

$$\begin{aligned} RE_{KSEW} = {} & \ln\left(\frac{ab\lambda\alpha\beta_1^{\lambda_1}}{a_1b_1\lambda_1\alpha_1\beta^\lambda}\right) + (\lambda-\lambda_1)E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right) - \frac{1}{\beta^\lambda}E\left(\tan^\lambda\left(\frac{\theta}{2}\right)\right) + \frac{1}{\beta_1^{\lambda_1}}E\left(\tan^{\lambda_1}\left(\frac{\theta}{2}\right)\right) \\ & +(a\alpha-1)E\left(\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right)\right) - (a_1\alpha_1-1)E\left(\ln\left(1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right)\right) \\ & +(b-1)E\left(\ln\left(1-\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^\lambda}\right)^{a\alpha}\right)\right) - (b_1-1)E\left(\ln\left(1-\left(1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right)^{a_1\alpha_1}\right)\right) \end{aligned} \tag{33}$$

where the expectations with specified parameters can be computed from (25), (26), (28), and (29).

H. Parameter's maximum likelihood estimators

Consider $(\theta_1, \theta_2, \dots, \theta_n)$ a random sample of size n follows $KSEW$ distribution with parameters $a, b, \lambda, , \alpha, \beta$. The maximum likelihood (ML) estimates of the five parameters can be obtained respectively by solving numerically the nonlinear equations $\frac{\partial \ln(L(\theta)_{KSEW})}{\partial a} = 0, \frac{\partial \ln(L(\theta)_{KSEW})}{\partial b} = 0, \frac{\partial \ln(L(\theta)_{KSEW})}{\partial \lambda} = 0, \frac{\partial \ln(L(\theta)_{KSEW})}{\partial \alpha} = 0, \frac{\partial \ln(L(\theta)_{KSEW})}{\partial \beta} = 0,$ $\ln(L(\theta)_{KSEW})$ represents the natural logarithm likelihood function of the pdf in (6) that is given by

$$\begin{aligned} \ln(L(\theta)_{KSEW}) = {} & n\ln\left(\frac{ab\lambda\alpha}{2\beta^\lambda}\right) + (\lambda-1)\sum_{i=1}^{n}\ln\left(\tan\left(\frac{\theta_i}{2}\right)\right) - \frac{1}{\beta^\lambda}\sum_{i=1}^{n}\tan^\lambda\left(\frac{\theta_i}{2}\right) \\ & +(a\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta_i}{2}\right)\right)^\lambda}\right) \\ & +(b-1)\sum_{i=1}^{n}\ln\left(1-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta_i}{2}\right)\right)^\lambda}\right]^{a\alpha}\right) + 2\sum_{i=1}^{n}\ln\left(\sec\left(\frac{\theta_i}{2}\right)\right) \end{aligned} \tag{34}$$

III. SIMULATION STUDY

A simulation study is conducted to perceive empirically the behaviors of the ML estimators of the $KSEW$ parameters for different default values of the parameters, $a = 0.8; b, \lambda, \alpha, \gamma = 0.8, 2.8$; $\beta = 0.8, 2.8$ and $\beta = 1.6, 5.6$ where $\beta = \gamma/v$ and

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$v = 1, 0.5$ and sample sizes $(n = 15, 30, 60, 120)$. With the run size of experiment $r = 1000$, the $KSEW$ random variable is generated by using the simulated formula in (17). The mean square error (MSE) has been utilized as a comparison criterion, where

$$MSE(\hat{\delta}) = \frac{1}{1000}\sum_{r=1}^{1000}\left(\hat{\delta}_r - \delta\right)^2;\ \delta = a, b, \lambda, \alpha \text{ or } \beta \qquad (62)$$

All default values and simulation results are listed in Tables I and II.

Table 1. The MSE values of the ML estimates of $KSEW$ distribution for different cases with $a = 0.8; b, \lambda, \alpha, \beta = 0.8, 2.8$.

Default Values					Sample Size	MSE values				
a	b	λ	α	β	n	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
0.8	0.8	0.8	0.8	0.8	15	0.0837	0.0496	0.1425	0.0617	0.0778
					30	0.0654	0.0331	0.0832	0.0487	0.0581
					60	0.0430	0.0184	0.0425	0.0294	0.0411
					120	0.0282	0.0123	0.0213	0.0197	0.0257
			2.8	2.8	15	0.1265	0.0857	0.0820	0.1743	0.3986
					30	0.0748	0.0522	0.0393	0.1434	0.3164
					60	0.0366	0.0289	0.0172	0.0997	0.2050
					120	0.0181	0.0160	0.0081	0.0682	0.1440
		2.8	0.8	0.8	15	0.0859	0.0928	0.5895	0.0947	0.0155
					30	0.0569	0.0633	0.4317	0.0623	0.0089
					60	0.0329	0.0365	0.2384	0.0284	0.0048
					120	0.0192	0.0229	0.1433	0.0174	0.0025
			2.8	2.8	15	0.2315	0.0714	0.4617	0.2625	0.1419
					30	0.1130	0.0433	0.2821	0.1746	0.0823
					60	0.0492	0.0218	0.1434	0.1088	0.0404
					120	0.0227	0.0139	0.0795	0.0844	0.0246
	2.8	0.8	0.8	0.8	15	0.0524	0.1399	0.1655	0.0391	0.1247
					30	0.0391	0.0950	0.0923	0.0297	0.0838
					60	0.0295	0.0639	0.0554	0.0242	0.0541
					120	0.0239	0.0510	0.0379	0.0163	0.0388
			2.8	2.8	15	0.0412	0.1303	0.1138	0.1522	0.3603
					30	0.0254	0.0935	0.0522	0.1091	0.2937
					60	0.0134	0.0688	0.0192	0.0707	0.1638
					120	0.0078	0.0479	0.0099	0.0436	0.1056
		2.8	0.8	0.8	15	0.0457	0.1038	0.2983	0.0372	0.0221
					30	0.0246	0.0764	0.2237	0.0208	0.0111
					60	0.0134	0.0668	0.1409	0.0135	0.0060
					120	0.0087	0.0444	0.0927	0.0108	0.0031
			2.8	2.8	15	0.1044	0.1147	0.4445	0.2083	0.1013
					30	0.0451	0.0768	0.3194	0.1322	0.0589
					60	0.0276	0.0585	0.1798	0.0802	0.0345
					120	0.0151	0.0470	0.0967	0.0450	0.0192

Table 2. The MSE values of the ML estimates of $KSEW$ distribution for different cases with $a = 0.8$; $b, \lambda, \alpha = 0.8, 2.8$; $\beta = 1.6, 5.6$.

Default Values					Sample Size	MSE values				
a	b	λ	α	β	n	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
0.8	0.8	0.8	0.8	1.6	15	0.1194	0.0896	0.1828	0.0907	0.2090
					30	0.0814	0.0628	0.0950	0.0651	0.1318
					60	0.0436	0.0344	0.0421	0.0373	0.0839
					120	0.0221	0.0191	0.0183	0.0243	0.0497
			2.8	5.6	15	0.0919	0.0861	0.0501	0.2055	0.6536
					30	0.0413	0.0510	0.0232	0.1691	0.5374
					60	0.0251	0.0360	0.0112	0.1108	0.3835
					120	0.0131	0.0248	0.0065	0.0975	0.2736
		2.8	0.8	1.6	15	0.0991	0.1018	0.6782	0.1044	0.0566
					30	0.0588	0.0667	0.4530	0.0598	0.0296
					60	0.0380	0.0429	0.2388	0.0329	0.0166
					120	0.0224	0.0252	0.1497	0.0203	0.0095
			2.8	5.6	15	0.2269	0.0861	0.5219	0.3498	0.3455
					30	0.0855	0.0409	0.2674	0.1863	0.1887
					60	0.0340	0.0291	0.1107	0.1391	0.1088
					120	0.0121	0.0145	0.0456	0.1082	0.0628
	2.8	0.8	0.8	1.6	15	0.0540	0.2514	0.1941	0.0385	0.2990
					30	0.0423	0.1876	0.1126	0.0335	0.1692
					60	0.0323	0.1317	0.0604	0.0248	0.1168
					120	0.0273	0.0902	0.0362	0.0174	0.0767
			2.8	5.6	15	0.0312	0.3153	0.1299	0.1808	0.6235
					30	0.0199	0.2041	0.0500	0.1077	0.4805
					60	0.0118	0.1364	0.0158	0.0593	0.2690
					120	0.0057	0.0871	0.0068	0.0337	0.1855
		2.8	0.8	1.6	15	0.0422	0.1344	0.3438	0.0393	0.0752
					30	0.0242	0.1183	0.3161	0.0280	0.0392
					60	0.0116	0.0720	0.1580	0.0162	0.0207
					120	0.0073	0.0610	0.1032	0.0117	0.0103
			2.8	5.6	15	0.0558	0.1998	0.7335	0.2738	0.2839
					30	0.0322	0.1331	0.4157	0.1854	0.1768
					60	0.0212	0.1000	0.1970	0.1097	0.1060
					120	0.0117	0.0668	0.0837	0.0392	0.0511

The results of tables 1 and 2, we can observe that:

With low default values of parameters b,λ,α,β for n=15,30 and 60, the most accurate is always associated with parameter b. Also, this result is true with the low value of b and high values of λ,α,β.

With high values of b,λ,α,β for n=30,60 and 120, the most accurate is always associated with parameter a. With low values of b,λ and high values of α,β, the most accurate is always associated with parameter λ.

With low values of b,α,β and high λ, the most accurate is always associated with parameter β. While, with high values of b,α,β and low λ, the most accurate is always associated with parameters a and λ respectively.

With high value of b and low values of λ,α,β, the most accurate is always associated with parameters α and a respectively.

The MSE values decrease with increasing sample size. This result is consistent with statistical theory.

As the default values of parameters b,λ,α, and β increase separately, there is always a simultaneous growth of MSE values of that parameters.

IV. CONCLUSION

More Transformed circular or semicircular distribution is a probability distribution that assigns probabilities or probability densities to different directions and revolves around a unit circle. Such distributions have proven to be extremely useful in modeling angular or circular data that arises in a variety of natural phenomena. This paper is focused on composing the Kumaraswamy-G family with semicircular exponentiated Weibull distribution to create a new transformed distribution called Kumaraswamy Semicircular Exponentiated Weibull (KSEW). The most important statistical properties are presented. A simulation study is conducted to evaluate the behavior of the ML parameters' estimates, which provides grounds for optimism about the distribution's stability and flexibility in practical applications. On the other side, when parameters a=b=1, the KSEW tends to SEW, which indicates that KSEW can be considered as a generalization of SEW distribution.

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