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Research Article

Using the Degree $r \in [0, 1]$ in Defining Rough Fuzzy Sets

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This paper introduced a new definition of rough fuzzy sets based on a fuzzy ideal ℓ defined on a fuzzy approximation space (X, R). This definition considered the degree $r \in [0, 1]$ in defining the after sets and the before sets of the arbitrary fuzzy relation R on the finite set of objects X. It is shown that this new type of roughness is a generalization to many of previous definitions in the fuzzy case and also in the ordinary case. As an application, it is given a rough fuzzy-connected set notion depending on some degree $r \in [0, 1]$.

1. Introduction

In [1], Pawlak initiated the notion of rough sets related with an equivalence relation R on a finite set X. Subsequently, researchers explored rough sets in the context of arbitrary relations [2–6], as well as investigating the generated interior and closure operators of rough sets and the existence of a generated topology on X [7–9]. Some researchers also studied rough fuzzy sets depending on previous notions and notations [10–15]. Others explored the use of fuzzy ideals [16] in fuzzy topological spaces [17, 18] and fuzzy approximation spaces [19] to generalize rough sets. Kozae et al. [7] defined general rough sets using the intersection of after and before sets of an arbitrary relation R on X, while Kandil et al. [4] used the notion of an ideal to define rough sets in the ordinary case. When the relation was symmetric, the definition in [4] became a generalization of the definitions in [2, 5, 7]. In [20], the authors used an arbitrary relation R on X and an ideal ℓ on X to define a generalized form of ordinary rough sets, which had a better accuracy value.

The motivation of this paper is coming from that the topology defined by Šostak is more generalized than the topology defined by Chang. That is, constructing structures and topological notions in sense of Šostak is a good extension to those notions. Rough sets were first depending on one parameter, the ordinary relation or the fuzzy relation defined on the universe set X. After many research works, Kandil [4] and Kozae [7] used a second parameter, the ordinary ideal or the fuzzy ideal on X, and their definitions was generalizations of the previous rough sets. But all these definitions are given in sense of Chang, which means roughness without any degree. From that point, we add here a third parameter, the degree $r \in [0, 1]$, and thus define rough sets in sense of Šostak as a generalization of all definitions given before. Neglecting the degrees $r \in [0, 1]$, we are restricted to the definition of Kozae [7], which was a generalization of the definition of Kandil [4], which again was a generalization of the previous basic definitions. From that point, this paper will cure the gap between the roughness in sense of Chang and the roughness in sense of Šostak.

In this paper, we propose a new definition of r-lower and r-upper fuzzy sets of r-rough fuzzy sets, which depend on the following three parameters: a fuzzy ideal ℓ on X, an arbitrary fuzzy relation R on X, and a degree $r \in [0, 1]$. Each parameter affects the roughness of a fuzzy set in I^X. This definition generalizes and fuzzifies the definitions introduced in [1–5, 7], and in the crisp case, it is identical to the definitions in [1–5, 7]. We also provide a more accurate measure of roughness and several interpreted examples in this paper. As an application of this new definition of rough fuzzy sets and as a generalization of connectedness in fuzzy topological spaces [21], we introduce the concept of rough fuzzy r-connectedness. The authors in [22] concentrated on giving a comparison between using minimal neighborhoods and maximal neighborhoods in defining rough fuzzy sets. In [22], the study was based on fuzzy sets in sense of Chang [23]. Since the fuzzy sets in sense of Šostak [24] paid attention for the degrees $r \in I$ and it was more general than the definition of Chang, then we define in this paper the roughness with some degree for fuzzy sets in a new style.

Throughout the paper, let X be a finite set and I is the closed interval [0, 1]. Denote for (0, 1] by I_0. I^X refers to all fuzzy subsets in X, and $\xi^c(x) = 1 - \xi(x)\ \forall x \in X, \forall \xi \in I^X$. A fuzzy point x_t at $t \in I$ is defined by $x_t(x) = t$ and $x_t(z) = 0\ \forall z \neq x$. A constant fuzzy set $\overline{t}$ for any $t \in I$ is defined by $\overline{t}(x) = t\ \forall x \in X$. Infimum and supremum of a fuzzy set $\xi \in I^X$ are given as $\inf \xi = \wedge_{x\in X}\xi(x)$ and $\sup\ \xi = \vee_{x\in X}\xi(x)$. If $f\colon X \longrightarrow Y$ is a mapping and $\zeta \in I^X, \nu \in I^Y$, then

$$\begin{aligned}(f(\zeta))(y) &= \bigvee_{x\in f^{-1}(y)} \zeta(x), \quad \forall y \in Y,\\ f^{-1}(\nu) &= (\nu \circ f).\end{aligned} \tag{1}$$

Recall that the fuzzy difference between two fuzzy sets was defined in [17] by

$$(\xi \,\overline{\wedge}\, \zeta) = \begin{cases}\overline{0}, & \text{if } \xi \leq \zeta,\\ \xi \wedge \zeta^c, & \text{otherwise.}\end{cases} \tag{2}$$

In [16], a map $\ell\colon I^X \longrightarrow I$ is called a fuzzy ideal (F-ideal) on X if it satisfies

(1) $\ell(\overline{0}) = 1$,

(2) $\xi \leq \zeta \Rightarrow \ell(\xi) \geq \ell(\zeta)$ for all $\xi, \zeta \in I^X$,

(3) $\ell(\xi \vee \zeta) \geq \ell(\xi) \wedge \ell(\zeta)$ for all $\xi, \zeta \in I^X$.

The triple (X, R, ℓ) is called a fuzzy ideal approximation space (FIA space). Define the fuzzy ideal ℓ° as an F-ideal ℓ such that $\ell(\zeta) = 0\ \forall \zeta \neq \overline{0}$.

2. r-Lower, r-Upper, and r-Boundary Region Fuzzy Sets

Here, we define rough fuzzy sets in a fuzzy approximation space (FA space) (X, R) in a generalized form in Šostak sense. The definitions will be based on the degree $r \in I$ and the fuzzy relationship $R(x, y)$ between each pair of elements of X.

Definition 1. Let X be a finite set and $R\colon X \times X \longrightarrow I$ an arbitrary fuzzy relation on X. Then, for any $x \in X$, define the fuzzy sets $xR, Rx \in I^X$ as follows [6]:

$$xR(y) = R(x, y) \text{ and } Rx(y) = R(y, x), \quad \forall y \in X. \tag{3}$$

Definition 2. For each $x \in X$ and $r \in I$, define an after fuzzy set $r \vdash x \dashv\colon X \longrightarrow I$ as follows:

$$r \vdash x \dashv (y) = \begin{cases} r \vee xR(y), & \text{if } R(x, y) \geq 0.5, \quad r > 0,\\ r \wedge xR(y), & \text{if } R(x, y) < 0.5, \quad r > 0,\\ 0, & \text{if } r = 0.\end{cases} \tag{4}$$

Definition 3. For each $x \in X$ and $r \in I$, define a before fuzzy set $\vdash x \dashv r\colon X \longrightarrow I$ as follows:

$$\vdash x \dashv r(y) = \begin{cases} r \vee Rx(y), & \text{if } R(y, x) \geq 0.5, \quad r > 0,\\ r \wedge Rx(y), & \text{if } R(y, x) < 0.5, \quad r > 0,\\ 0, & \text{if } r = 0.\end{cases} \tag{5}$$

For all $r \in I_0$, $r \vdash x \dashv$ or $\vdash x \dashv r$ is representing the strength of the bond between an element x and any element in X so that their membership values are from 0.5 and up to 1 whenever $R(x, y)$ or $R(y, x) \geq 0.5$ and less than 0.5 and down to 0 whenever $R(x, y)$ or $R(y, x) < 0.5$. In particular, $1 \vdash x \dashv (y) = R(x, y)$ whenever $R(x, y) < 0.5$ and $1 \vdash x \dashv (y) = 1$ otherwise, $\vdash x \dashv 1(y) = R(y, x)$ whenever $R(y, x) < 0.5$ and $\vdash x \dashv 1(y) = 1$ otherwise. $0 \vdash x \dashv (y) = \vdash x \dashv 0(y) = 0\ \forall y \in X$, which means that $0 \vdash x \dashv = \vdash x \dashv 0 = \overline{0}\ \forall x \in X$. That is, according to the degree $r = 0$, we have empty fuzzy sets all time, and so we will not consider the case $r = 0$ in all the following results. For any $r \in I_0$, if R is reflexive, then $r \vdash x \dashv \wedge \vdash x \dashv r \geq x_1\ \forall x \in X$. That is, if R is reflexive, we ensure that for any $x \in X$ and any $r \in I_0$, we have nonempty fuzzy sets $r \vdash x \dashv, \vdash x \dashv r$, and these fuzzy sets contain at least the fuzzy point x_1. If R is symmetric, then $r \vdash x \dashv = \vdash x \dashv r$ for all $r \in I$ and will be denoted by $\vdash x \dashv^r$.

Definition 4. Define for any $a \in X$ and any $r \in I_0$, the fuzzy sets $\vdash a \dashv R,\ R \vdash a \dashv \in I^X$ as follows:

$$\begin{aligned}\vdash a \dashv R &= \bigwedge_{x\in X,\, r\vdash x\dashv(a)>0} r \vdash x \dashv,\\ R \vdash a \dashv &= \bigwedge_{x\in X,\, \vdash x\dashv r(a)>0} \vdash x \dashv r.\end{aligned} \tag{6}$$

For any $a \in X$, define $R \vdash a \dashv R\colon X \longrightarrow I$ as follows:

$$R \vdash a \dashv R = \vdash a \dashv R \wedge R \vdash a \dashv. \tag{7}$$

Definition 5. For every $x \in X$, define $\xi_*, \xi^* \in I^X$ of a fuzzy set $\xi \in I^X$ by

$$\xi_*(x) = \begin{cases} \left[\bigvee_{z\in X} R\vdash x \dashv R(x)\right]^c, & \text{if } \ell(R\vdash x\dashv R\wedge \xi^c) = 0 \text{ and } \ell(R\vdash x\dashv R\wedge \xi) = 0, \\ 1, & \text{if } \ell(R\vdash x\dashv R\wedge \xi^c) > 0, \\ 0, & \text{if } \ell(R\vdash x\dashv R\wedge \xi^c) = 0 \text{ and } \ell(R\vdash x\dashv R\wedge \xi) > 0, \end{cases} \tag{8}$$

$$\xi^*(x) = \begin{cases} \bigvee_{z\in X} R\vdash x \dashv R(x), & \text{if } \ell(R\vdash x\dashv R\wedge \xi) = 0 \text{ and } \ell(R\vdash x\dashv R\wedge \xi^c) = 0, \\ 0, & \text{if } \ell(R\vdash x\dashv R\wedge \xi) > 0, \\ 1, & \text{if } \ell(R\vdash x\dashv R\wedge \xi) = 0 \text{ and } \ell(R\vdash x\dashv R\wedge \xi^c) > 0. \end{cases} \tag{9}$$

The roughness, with degree $r \in I_0$, of a fuzzy set $\xi \in I^X$ is defined by

$$\begin{aligned} \xi_R &= \xi \wedge \xi_*, \\ \xi^R &= \xi \vee \xi^*. \end{aligned} \tag{10}$$

ξ_R is the r-lower fuzzy set of ξ and ξ^R is the r-upper fuzzy set of ξ. The r-boundary region fuzzy set of ξ is ξ^B given by $\xi^B = \xi^R \bar{\wedge} \xi_R$. The pair (X, R) will be called a rough fuzzy approximation (RFA) space.

Definition 6. For every rough fuzzy set $\xi \in I^X$, define the accuracy fuzzy set $\alpha(\xi) \in I^X$, for all $x \in X$, by the following:

$$\alpha(\xi)(x) = \begin{cases} \left(\xi^B(x)\right)^c, & \text{if } \xi^R \nleq \xi_R, \\ 1, & \text{otherwise,} \end{cases} \tag{11}$$

and moreover, the accuracy value of the rough fuzzy set ξ is given by $\mathrm{Inf}(\alpha(\xi))$.

If we neglected the degree $r \in I$ along the paper, then we do not need equations (4) and (5) and only use equation (3) in the previous Definitions 4–6 since $r\vdash x\dashv \equiv xR$ and $\vdash x \dashv r \equiv Rx$, $\forall x \in X$.

Whenever ξ^R is so that $\xi^R \leq \xi_R$, we get that $\xi = \xi_R = \xi^R$ and then $\xi^B = \overline{0}$ and $\mathrm{Inf}(\alpha(\xi)) = 1$. If $\xi_R = \overline{0}$ and $\xi^R = \overline{1}$, then $\xi^B = \overline{1}$ and $\mathrm{Inf}(\alpha(\xi)) = 0$.

Otherwise, $\xi^B = \xi^R \wedge (\xi_R)^c$ and $0 < \mathrm{Inf}(\alpha(\xi)) < 1$. That is, the largest r-boundary fuzzy set is associated with the lowest accuracy value and the converse is true. If $\mathrm{Inf}(\alpha(\xi)) = 1$, then ξ is crisp with respect to R ($\xi_R = \xi^R$ and ξ is precise with respect to R). If $\mathrm{Inf}(\alpha(\xi)) = 0$, then ξ is totally rough with respect to R. Moreover, if $0 < \mathrm{Inf}(\alpha(\xi)) < 1$, then ξ is rough with respect to R.

Lemma 7. *Let R be a fuzzy relation on X, $r \in I_0$, ℓ an F-ideal on X, and $\xi, \zeta \in I^X$. Then, the following properties hold:*

(1) $\xi^* = ((\xi^c)_*)^c$, $\xi_* = ((\xi^c)^*)^c$,
(2) $\overline{0}^* = \overline{0}$ and $\overline{1}_* = \overline{1}$,
(3) $\xi \leq \zeta$ implies that $\xi_* \leq \zeta_*$ and $\xi^* \leq \zeta^*$,
(4) $(\xi \wedge \zeta)^* \leq \xi^* \wedge \zeta^*$,
(5) $(\xi \vee \zeta)_* \geq \xi_* \vee \zeta_*$,
(6) $(\xi \wedge \zeta)_* \leq \xi_* \wedge \zeta_*$,
(7) $(\xi \vee \zeta)^* \geq \xi^* \vee \zeta^*$,
(8) If $\ell(\xi) > 0$, then $\xi^* = \overline{0}$. Moreover, if $\ell(\xi^c) > 0$, then $\xi_* = \overline{1}$.

Proof. For (1), it is clear that $(\xi^*)^c = (\xi^c)_*$, $(\xi_*)^c = (\xi^c)^*$, and thus $\xi^* = ((\xi^c)_*)^c$ and $\xi_* = ((\xi^c)^*)^c$.

For (2), we have $(\overline{0})^*(x) = 0$, $\forall x \in X$ from equation (9) and $(\overline{1})_*(x) = 1$, $\forall x \in X$ from equation (8), and then $\overline{0}^* = \overline{0}$ and $\overline{1}_* = \overline{1}$.

For (3), it is proved from the definition of the F-ideal and equations (8) and (9).

For (4) and (5), we get it directly using the result in (3).

For (6), we get that $\xi_* \wedge \zeta_* \geq (\xi \wedge \zeta)_*$ directly.

For (7), we get that $\xi^* \vee \zeta^* \leq (\xi \vee \zeta)^*$ directly.

For (8), since $\ell(\xi) > 0$ implies that $\ell(R\vdash x\dashv R\wedge \xi) > 0$ and thus $\xi^*(x) = 0$, $\forall x \in X$. Hence, $\xi^* = \overline{0}$. Similarly, $\ell(\xi^c) > 0$ implies that $\ell(R\vdash x\dashv R\wedge \xi^c) > 0$ and thus $\xi_*(x) = 1$, $\forall x \in X$. Hence, $\xi_* = \overline{1}$.

Note that if we have the trivial F-ideal ℓ for which $\ell(\overline{1}) = 1$, then $\xi_* = \overline{1}$ and $\xi^* = \overline{0}$, and hence $\xi_R = \xi^R = \xi$, and, therefore, any fuzzy set has accuracy value $\mathrm{Inf}(\alpha(\xi)) = 1$. □

Remark 8. Let R be a fuzzy relation on X, ℓ be an F-ideal on X, and $\xi, \zeta \in I^X$. Then, the following results hold in general.

(1) $\xi \nleq \xi_* \nleq \xi^* \nleq \xi$ and $\xi \nleq \xi^* \nleq \xi_* \nleq \xi$,
(2) $(\xi \wedge \zeta)^* \ngeq \xi^* \wedge \zeta^*$ and $(\xi \wedge \zeta)_* \ngeq \xi_* \wedge \zeta_*$,
(3) $(\xi \vee \zeta)_* \nleq \xi_* \vee \zeta_*$ and $(\xi \vee \zeta)^* \nleq \xi^* \vee \zeta^*$,
(4) $\xi^* = \overline{0} \nRightarrow \ell(\xi) > 0$.
(5) $\xi_* = \overline{1} \nRightarrow \ell(\xi^c) > 0$.

The following example proves the results in Remark 8.

Example 1. Let $r = 0.4 \in I_0$ and R be a fuzzy relation on a set $X = \{a, b, c, d\}$ as follows.

R	a	b	c	d
a	0	0.2	1	0.5
b	0.6	0	0.8	0.5
c	1	0.5	0.6	0.6
d	0.9	0.6	1	1

$0.4\vdash a\dashv=\{0,0.2,1,0.5\}$, $0.4\vdash b\dashv=\{0.6,0,0.8,0.5\}$, $0.4\vdash c\dashv=\{1,0.5,0.6,0.6\}$, $0.4\vdash d\dashv=\{0.9,0.6,1,1\}$ and $\vdash a\dashv 0.4=\{0,0.6,1,0.9\}$, $\vdash b\dashv 0.4=\{0.2,0,0.5,0.6\}$, $\vdash c\dashv 0.4=\{1,0.8,0.6,1\}$, $\vdash d\dashv 0.4=\{0.5,0.5,0.6,1\}$. Then, $\vdash a\dashv R=\{0.6,0,0.6,0.5\}$, $\vdash b\dashv R=\{0,0.2,0.6,0.5\}$, $\vdash c\dashv R=\{0,0,0.6,0.5\}$, $\vdash d\dashv R=\{0,0,0.6,0.5\}$ and $R\vdash a\dashv=\{0.2,0,0.5,0.6\}$, $R\vdash b\dashv=\{0,0.5,0.6,0.9\}$, $R\vdash c\dashv=\{0,0,0.5,0.6\}$, $R\vdash d\dashv=\{0,0,0.5,0.6\}$, and then $R\vdash a\dashv R=\{0.2,0,0.5,0.5\}$, $R\vdash b\dashv R=\{0,0.2,0.6,0.5\}$, $R\vdash c\dashv R=\{0,0,0.5,0.5\}$, $R\vdash d\dashv R=\{0,0,0.5,0.5\}$.

(1) Consider an F-ideal ℓ on X so that $\ell(\nu)>0$, $\forall\nu\leq\overline{0.4}$; otherwise, $\ell(\nu)=0$. Then, we compute ξ_*,ξ^* for a fuzzy set $\xi=\{0.1,0.9,0.3,0.6\}$ as follows: $\xi_*=\{0.8,0.8,0.4,0.5\}$, $\xi^*=\{0.2,0.2,0.6,0.5\}$. Hence, $\xi_*\leq\xi^*\leq\xi\leq\xi_*$\\and $\xi^*\leq\xi_*\leq\xi\leq\xi^*$ and thus (1) holds.

(2) Let $\zeta=\{0.6,0.2,0.9,0.2\}$, we get that $\zeta^*=\{0.2,0.2,0.6,0.5\}$, $\zeta_*=\{0.8,0.8,0.4,0.5\}$, and so $\zeta^*=\xi^*=\{0.2,0.2,0.6,0.5\}$, $\zeta_*=\xi_*=\{0.8,0.8,0.4,0.5\}$. For $\xi\wedge\zeta=\{0.1,0.2,0.3,0.2\}$, we get that $(\xi\wedge\zeta)^*=\overline{0}$. Hence, $\xi^*\wedge\zeta^*=\{0.2,0.2,0.6,0.5\}\leq\overline{0}=(\xi\wedge\zeta)^*$. Also, we get that $(\xi\wedge\zeta)_*=\overline{0}\geq\{0.8,0.8,0.4,0.5\}=\xi_*\wedge\zeta_*$. Thus, (2) is proved.

(3) Again, for $\xi\vee\zeta=\{0.6,0.9,0.9,0.6\}$, we get that $(\xi\vee\zeta)_*=\overline{1}$. Hence, $(\xi\vee\zeta)_*=\overline{1}\leq\xi_*\vee\zeta_*=\{0.8,0.8,0.4,0.5\}$. Also, we get that $(\xi\vee\zeta)^*=\overline{1}\leq\{0.2,0.2,0.6,0.5\}=\xi^*\vee\zeta^*$. Thus, (3) is proved.

(4) Now define an F-ideal ℓ on X so that $\ell(\nu)>0$, $\forall\nu\leq\overline{0.5}$; otherwise, $\ell(\nu)=0$, and consider the same fuzzy set $\xi=\{0.1,0.9,0.3,0.6\}$. We compute ξ_*,ξ^* as follows: $\xi^*=\{0,0,0,0\}=\overline{0}$, $\xi_*=\{1,0,1,1\}$. That is, $\xi^*=\overline{0}$ while $\xi\notin\ell$. That is, (4) is proved.

(5) Moreover, if ℓ is defined on X so that $\ell(\nu)>0$, $\forall\nu\leq\overline{0.6}$; otherwise, $\ell(\nu)=0$, then $\xi_*=\overline{1}$ while $\xi^c\notin\ell$ and thus (5) is proved.

Lemma 9. *The r-lower and r-upper fuzzy sets of fuzzy sets satisfy the following properties:*

(1) $\xi_R\leq\xi\leq\xi^R$,
(2) $\overline{0}_R=\overline{0}^R=\overline{0}$ *and* $\overline{1}_R=\overline{1}^R=\overline{1}$,
(3) $(\xi\vee\zeta)_R\geq\xi_R\vee\zeta_R\ \forall\xi,\zeta\in I^X$,
(4) $(\xi\wedge\zeta)^R\leq\xi^R\wedge\zeta^R\ \forall\xi,\zeta\in I^X$,
(5) $\xi\leq\zeta$ *implies that* $\xi_R\leq\zeta_R$ *and* $\xi^R\leq\zeta^R\forall\xi,\zeta\in I^X$,
(6) $(\xi\wedge\zeta)_R\leq\xi_R\wedge\zeta_R\forall\xi,\zeta\in I^X$,
(7) $(\xi\vee\zeta)^R\geq\xi^R\vee\zeta^R\forall\xi,\zeta\in I^X$,
(8) $(\xi^R)^c=(\xi^c)_R$ *and* $(\xi_R)^c=(\xi^c)^R$
(9) $(\xi_R)^R\geq\xi_R\geq(\xi_R)_R$
(10) $(\xi^R)_R\leq\xi^R\leq(\xi^R)^R$

Proof. From (10), Lemma 7, we get easily the proof of all these results. □

Remark 10. As in the usual case, if R is a reflexive fuzzy relation on X, then $\xi_*\leq\xi\leq\xi^*,\forall\xi\in I^X$. In this case, the equality holds in both of (6) and (7) in Lemma 7, and thus the equality holds in both of (6) and (7) in Lemma 9.

Moreover, as in the usual case, if R is a reflexive and a transitive fuzzy relation, then $(\xi_R)_R=\xi_R$ and $(\xi^R)^R=\xi^R$.

If we consider R a reflexive fuzzy relation on X, then a fuzzy pretopology τ_R on the RFA space (X,R) is generated by the following:

$$\tau_R=\left\{\nu\in I^X\colon\ \nu=\nu_R\right\}\text{ or }\tau_R=\left\{\nu\in I^X\colon\ \nu^c=\left(\nu^c\right)^R\right\}.\qquad(12)$$

That is, the condition $(\xi_R)_R=\xi_R,\forall\xi\in I^X$ is not satisfied.

If we consider R a reflexive and a transitive fuzzy relation on X, then a fuzzy topology τ_R is generated on the RFA space (X,R) by equation (12) as well. That is, the condition $(\xi_R)_R=\xi_R,\forall\xi\in I^X$ is satisfied.

In the following, using only the after fuzzy sets (using before fuzzy sets is similar), it will be defined a weaker definition than Definition 5.

Definition 11. For every $x \in X$ and $r \in I_0$, define $\xi_{**}, \xi^{**} \in I^X$ of $\xi \in I^X$ by

$$\xi_{**}(x) = \begin{cases} \left(\bigvee_{z\in X} \vdash z \dashv R(x)\right)^c, & \text{if } \ell(\vdash x \dashv R \wedge \xi^c) = 0 \text{ and } \ell(\vdash x \dashv R \wedge \xi) = 0, \\ 1, & \text{if } \ell(\vdash x \dashv R \wedge \xi^c) > 0, \\ 0, & \text{if } \ell(\vdash x \dashv R \wedge \xi^c) = 0 \text{ and } \ell(\vdash x \dashv R \wedge \xi) > 0, \end{cases} \tag{13}$$

$$\xi^{**}(x) = \begin{cases} \bigvee_{z\in X} \vdash x \dashv R(x) & \text{if } \ell(\vdash x \dashv R \wedge \xi) = 0 \text{ and } \ell(\vdash x \dashv R \wedge \xi^c) = 0, \\ 0 & \text{if } \ell(\vdash x \dashv R \wedge \xi) > 0, \\ 1 & \text{if } \ell(\vdash x \dashv R \wedge \xi) = 0 \text{ and } \ell(\vdash x \dashv R \wedge \xi^c) > 0. \end{cases} \tag{14}$$

The roughness of a fuzzy set $\xi \in I^X$ is defined by

$$\underline{\xi} = \xi \wedge \xi_{**} \text{ and } \overline{\xi} = \xi \vee \xi^{**}, \tag{15}$$

where $\underline{\xi}$ is the r-lower fuzzy set of ξ and $\overline{\xi}$ is the r-upper fuzzy set of ξ.

The r-boundary region fuzzy set $B(\xi)$ of ξ is given by $B(\xi) = \overline{\xi} \,\overline{\wedge}\, \underline{\xi}$.

All the results given in the section are satisfied exactly, only the main difference is coming from equation (7). That is, $R \vdash x \dashv R \leq \vdash x \dashv R, \forall x \in X$. Hence, Definition 5 gives us an r-boundary region fewer than that of Definition 11.

Theorem 12. *It is clear by definitions the following:*

(1) $\xi_{**} \leq \xi_*$, *and then* $\underline{\xi} \leq \xi_R$,
(2) $\xi^* \leq \xi^{**}$, *and then* $\xi^R \leq \overline{\xi}$.

Example 2. As in Example 1, where $r = 0.4$ and $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.4}$, we got for $\xi = \{0.1, 0.9, 0.3, 0.6\}$ that $\xi^* = \{0.2, 0.2, 0.6, 0.5\}$, $\xi_* = \{0.8, 0.8, 0.4, 0.5\}$, and then $\xi^R = \{0.2, 0.9, 0.6, 0.6\}$, $\xi_R = \{0.1, 0.8, 0.3, 0.5\}$. That is, $\xi^B = \{0.2, 0.2, 0.6, 0.5\}$.

From equation (11), we get that $\alpha(\xi) = \{0.9, 0.9, 0.7, 0.9\}$, and then $\operatorname{Inf}(\alpha(\xi)) = 0.7$.

If we used Definition 11, we get $\xi^{**} = \{0.6, 0.2, 0.6, 0.5\}$, $\xi_{**} = \{0.4, 0.8, 0.4, 0.5\}$, and then $\overline{\xi} = \{0.6, 0.9, 0.6, 0.6\}$, $\underline{\xi} = \{0.1, 0.8, 0.3, 0.5\}$. Thus, $B(\xi) = \{0.6, 0.2, 0.6, 0.5\}$ and $\xi^B \leq B(\xi)$. Hence, the boundary region of Definition 5 is better than the boundary region of Definition 11.

In case of $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.5}$, we get that $\xi_* = \{1, 0, 1, 1\}$, $\xi^* = \overline{0}$, and then $\xi^R = \{0.1, 0.9, 0.3, 0.6\}$, $\xi_R = \{0.1, 0, 0.3, 0.6\}$. That is, $\xi^B = \{0.1, 0.9, 0.3, 0.4\}$. Moreover, $\alpha(\xi) = \{1, 0.1, 1, 1\}$ and $\operatorname{Inf}(\alpha(\xi)) = 0.1$.

Note that if we discussed the crisp case or the classical case ($r \in \{0, 1\}$), then we have R as a classical binary relation on X that has at least $R(x, y) = 1$ for some $x, y \in X$ and an F-ideal $\ell = \{\overline{0}\}$. Hence, the first branch in equation (8) goes to zero, and the first branch in equation (9) goes to one. Thus, Definitions 5 and 11 will be as follows:

$$\begin{aligned} \xi_*(x) &= \begin{cases} 1, & \text{if } R \vdash x \dashv R \wedge \xi^c = \overline{0}, \\ 0, & \text{otherwise,} \end{cases} \\ \xi^*(x) &= \begin{cases} 1, & \text{if } R \vdash x \dashv R \wedge \xi \neq \overline{0}, \\ 0, & \text{otherwise,} \end{cases} \\ \xi_{**}(x) &= \begin{cases} 1, & \text{if } \vdash x \dashv R \wedge \xi^c = \overline{0}, \\ 0, & \text{otherwise,} \end{cases} \\ \xi^{**}(x) &= \begin{cases} 1, & \text{if } \vdash x \dashv R \wedge \xi \neq \overline{0}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{16}$$

That means Definition 5 will be the same meaning of rough sets in the ordinary case using the intersection of after sets and before sets of a classical binary relation R on X. Also, Definition 11 will be the same meaning in the ordinary case only using after sets of the classical binary relation R on X.

Here, recall some definitions in the ordinary case, and we will see that Definition 5 is a generalization for all of them in the fuzzy case and also in the ordinary case under suitable restrictions. That is, we neglected the degree $r \in I$ in our definitions in the fuzzy case.

In the fuzzy case, we can define the fuzzification of the definitions of Pawlak [1], Yao [5], Allam [3], Kandil [4], and Kozae [7], respectively, in the following:

(1) Let R be a fuzzy equivalence relation on X, $\vdash x \dashv R; x \in X$ is a fuzzy set defined by $\vdash x \dashv R(y) = R(x, y)$. For any fuzzy subset A of X, the lower fuzzy approximation $\underline{A}$ and the upper fuzzy approximation $\overline{A}$ are defined so that

$$\begin{aligned} \underline{A}(x) > 0 &\Leftrightarrow \vdash x \dashv R \leq A, \\ \overline{A}(x) > 0 &\Leftrightarrow \vdash x \dashv R \wedge A \neq \overline{0}. \end{aligned} \tag{17}$$

(2) Let R be an arbitrary fuzzy relation on X. For any fuzzy subset A of X, the lower fuzzy approximation $\underline{A}$ and the upper fuzzy approximation $\overline{A}$ are defined so that

$$\begin{aligned} \underline{A}(x) > 0 &\Leftrightarrow xR \leq A, \\ \overline{A}(x) > 0 &\Leftrightarrow xR \wedge A \neq \overline{0}, \end{aligned} \tag{18}$$

where xR is the after fuzzy set of x defined by $xR(y) = R(x, y)$.

Moreover, Rx is the before fuzzy set of x defined by $Rx(y) = R(y, x)$.

(3) Let R be an arbitrary fuzzy relation on X, $<p>R$ is the infimum of all the after fuzzy sets xR containing p. Then, for any fuzzy subset A of X, the lower fuzzy approximation $\underline{A}$ and the upper fuzzy approximation $\overline{A}$ are defined so that

$$\begin{aligned} \underline{A}(x) > 0 &\Leftrightarrow \vdash x \dashv R \leq A, \\ \overline{A}(x) > 0 &\Leftrightarrow \vdash x \dashv R \wedge A \neq \overline{0}, \end{aligned} \tag{19}$$

where $\vdash p \dashv R$, for any $p \in X$, is defined by $\vdash p \dashv R = \wedge_{0<xR(p), x\in X} xR$. Moreover, $R \vdash p \dashv = \wedge_{0<xR(p), x\in X} Rx$.

(4) Let R be a reflexive fuzzy relation on X and L be an F-ideal on X. For any fuzzy subset A of X, the lower fuzzy approximation $\underline{A}$ and the upper fuzzy approximation $\overline{A}$ are defined so that

$$\underline{A}(x) > 0 \Leftrightarrow (\vdash x \dashv R \wedge A^c) \in L, \tag{20}$$

$$\overline{A}(x) > 0 \Leftrightarrow A(x) > 0, (\vdash x \dashv R \wedge A) \notin L. \tag{21}$$

(5) Let R be an arbitrary fuzzy relation on X. For any fuzzy subset A of X, the lower fuzzy approximation $\underline{A}$ and the upper fuzzy approximation $\overline{A}$ are defined so that

$$\begin{aligned} \underline{A}(x) > 0 &\Leftrightarrow R \vdash x \dashv R \leq A, \\ \overline{A}(x) > 0 &\Leftrightarrow R \vdash x \dashv R \wedge A \neq \bar{0}, \end{aligned} \tag{22}$$

where $R \vdash p \dashv R$ is defined by: $R \vdash p \dashv R = \vdash p \dashv R \wedge R \vdash p \dashv$.

The previous definitions are deduced as special cases for the roughness of a fuzzy set in a RFA space (X, R), from Definition 5 and equation (10).

Remark 13. From equations (8) and (9) and neglecting the degree $r \in I$, we get that

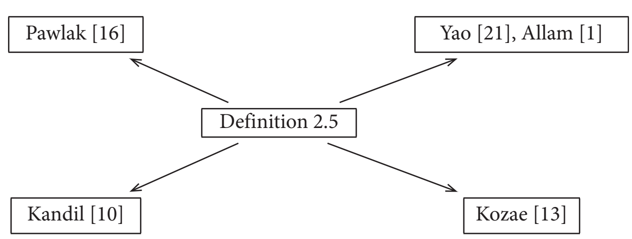
(1) if we have an equivalence fuzzy relation R and $\ell = \ell^{\circ}$ on X, then our definition will be the fuzzification of the main definition given by Pawlak in [1].

(2) if we have a symmetric fuzzy relation R on X and the F-ideal $\ell = \ell^{\circ}$ on X, then our definition will be the fuzzification of the definition given by Yao [5] and also the definition of Allam [2].

(3) if we have a reflexive and symmetric fuzzy relation R on X, then our definition will be the fuzzification of the definition given by Kandil [4].

(4) If we replaced the F-ideal ℓ on X with the F-ideal $\ell = \ell^{\circ}$, then our definition will be the fuzzification of the definition given by Kozae [7].

In the crisp case, we get exactly the definitions in the ordinary case as given in [1, 2, 4, 5, 7], respectively.



For a fuzzy set ξ in a FA space (X, R) where R is a reflexive fuzzy relation, we get a Č ech fuzzy interior operator int_R: $I^X \times I_0 \longrightarrow I_0$ and a Č ech fuzzy closure operator cl_R: $I^X \times I_0 \longrightarrow I_0$ as follows:

$$\begin{aligned} \mathrm{int}_R(\xi, r) &= \xi_R, \\ \mathrm{cl}_R(\xi, r) &= \xi^R. \end{aligned} \tag{23}$$

(That means $\mathrm{int}_R(\mathrm{int}_R(\xi, r)) \neq \mathrm{int}_R(\xi, r)$ and $\mathrm{cl}_R(\mathrm{cl}_R(\xi, r)) \neq \mathrm{cl}_R(\xi, r)$).

Also, from equation (15), we can define for any $\xi \in I^X$ a Č ech fuzzy interior operator I_R: $I^X \times I_0 \longrightarrow I_0$ and a Č ech fuzzy closure operator C_R: $I^X \times I_0 \longrightarrow I_0$ as follows:

$$\begin{aligned} I_R(\xi, r) &= \underline{\xi}, \\ C_R(\xi, r) &= \overline{\xi}. \end{aligned} \tag{24}$$

It is clear from Theorem 12 that the Č ech fuzzy interior operator int_R and the Č ech fuzzy closure operator cl_R have the following properties related with the Č ech fuzzy interior operator I_R and the Č ech fuzzy closure operator C_R:

$$\mathrm{int}_R(\xi, r) \geq \mathrm{I}_R(\xi, r) \text{ and } \mathrm{cl}_R(\xi, r) \leq \mathrm{C}_R(\xi, r), \quad \forall r \in I_0, \forall \xi \in I^X. \tag{25}$$

Note that the fuzzy operators cl_R, C_R of $\xi = \{0.1, 0.9, 0.3, 0.6\}$ in Example 1 (where $r = 0.4$, R was not reflexive) are not even Č ech fuzzy closure operators while both are computed as $\mathrm{cl}_R(\xi, r) = \xi^R = \{0.2, 0.9, 0.6, 0.6\} \leq C_R(\xi, r) = \overline{\xi} = \{0.6, 0.9, 0.6, 0.6\}$.

Considering R as a reflexive and transitive fuzzy relation, then we have a fuzzy interior operator and a fuzzy closure operator on (X, R) generating fuzzy topology τ_R as in equation (12). In this case, the usual properties of fuzzy interior and fuzzy closure operators are satisfied as follows.

Lemma 14. *The following conditions are satisfied.*

(1) $\mathrm{int}_R(\overline{0}, r) = \overline{0}$, $\mathrm{int}_R(\overline{1}, r) = \overline{1}$,

(2) $\mathrm{int}_R(\nu, r) \leq \nu, \forall r \in I_0, \nu \in I^X$,

(3) $\nu \leq \eta \Rightarrow \mathrm{int}_R(\nu, r) \leq \mathrm{int}_R(\eta, r), \forall r \in I_0, \nu, \eta \in I^X$,

(4) $\mathrm{int}_R((\nu \vee \eta), r) \geq \mathrm{int}_R(\nu, r) \vee \mathrm{int}_R(\eta, r)$, $\mathrm{int}_R((\nu \wedge \eta), r) = \mathrm{int}_R(\nu, r) \wedge \mathrm{int}_R(\eta, r)$, $\forall r \in I_0, \nu, \eta \in I^X$,

(5) $\mathrm{int}_R(\mathrm{int}_R(\nu, r)) = \mathrm{int}_R(\nu, r), \forall r \in I_0, \nu \in I^X$.

Proof. (1) and (2) are clear.

From (5) in Lemma 9, we get that (3) is satisfied.

According to Remark 10, and from (3) and (7) in Lemma 9, we get (4) is proved.

Also, by Remark 10 and from (9) in Lemma 9, we get that (5) is proved.

Thus, $\mathrm{int}_R(\xi, r)$ is the fuzzy interior of ξ in the RFA space (X, R) generating a fuzzy topology defined by

$$\varpi_R = \left\{\nu \in I^X \colon \nu = \mathrm{int}_R(\nu, r)\right\}. \tag{26}$$

Note that $I_R(\xi, r)$ of any $\xi \in I^X$ in the RFA space (X, R) with R as a reflexive and transitive fuzzy relation is also generating a fuzzy topology defined by

$$\omega_R = \left\{\nu \in I^X \colon \nu = \mathrm{I}_R(\nu, r)\right\}, \tag{27}$$

and thus this is coarser than that one generated by $\mathrm{int}_R(\xi, r)$. That is, $\omega_R \leq \varpi_R$.

Note that $\mathrm{cl}_R(\nu^R, r) = \mathrm{cl}_R(\nu, r), \forall r \in I_0, \nu \in I^X$, $\mathrm{int}_R(\nu_R, r) = \mathrm{int}_R(\nu, r), \forall r \in I_0, \nu \in I^X$, $\mathrm{int}_R(\nu^c, r) = (\mathrm{cl}_R(\nu, r))^c$ and $\mathrm{cl}_R(\nu^c, r) = (\mathrm{int}_R(\nu, r))^c \forall r \in I_0, \nu \in I^X$.

Similarly, where R is reflexive and transitive, we have the following. □

Lemma 15. *The fuzzy closure operator satisfy the following conditions:*

(1) $cl_R(\overline{0}, r) = \overline{0}$, $cl_R(\overline{1}, r) = \overline{1}$,

(2) $cl_R(\nu, r) \geq \nu, \forall r \in I_0, \nu \in I^X$,

(3) $\nu \leq \eta \Rightarrow cl_R(\nu, r) \leq cl_R(\eta, r), \forall r \in I_0, \nu, \eta \in I^X$,

(4) $cl_R((\nu \wedge \eta), r) \leq cl_R(\nu, r) \wedge cl_R(\eta, r)$, $cl_R((\nu \vee \eta), r) = cl_R(\nu, r) \vee cl_R(\eta, r), \forall r \in I_0, \nu, \eta \in I^X$,

(5) $cl_R(cl_R(\nu, r)) = cl_R(\nu, r), \forall r \in I_0, \nu \in I^X$.

Proof. Similar to Lemma 14.

Hence, from $\mathrm{cl}_R(\nu^c, r) = (\mathrm{int}_R(\nu, r))^c$, cl_R is a fuzzy closure operator generating the same fuzzy topology given above.

As an application of these generalized rough fuzzy sets, we discuss rough fuzzy-connected spaces with degree $r \in I_0$ using fuzzy closure operators where R is a reflexive and transitive fuzzy relation. □

3. r-Connectedness in Rough Fuzzy Approximation Spaces

Definition 16. Let (X, R) be a rough fuzzy approximation space (RFA space), $r \in I_0$, and ℓ an F-ideal on X. Then,

(1) The fuzzy sets $\zeta, \nu \in I^X$ are called rough fuzzy approximation r-separated (RF r-sep.) if $cl_R(\zeta, r) \wedge \nu = \zeta \wedge cl_R(\nu, r) = \overline{0}$.

(2) A fuzzy set $\eta \in I^X$ is called rough fuzzy approximation r-disconnected (RF r-disconnected) set if there exist RF r-sep. sets $\zeta, \nu \in I^X$ such that $\zeta \vee \nu = \eta$. A fuzzy set η is called rough fuzzy approximation r-connected (RF r-connected) if it is not RF r-disconnected. In other words, if there exist no RF r-sep. sets ζ, ν except $\zeta = \overline{0}$ or $\nu = \overline{0}$.

(3) A RFA space (X, R) is called RF r-disconnected space if there exist RF r-sep. sets $\zeta, \nu \in I^X$ such that $\zeta \vee \nu = \overline{1}$. A RFA space (X, R) is called RF r-connected if it is not RF r-disconnected.

Remark 17. Any two RF r-sep. sets ζ, ν in I^X with respect to the fuzzy closure operator C_R defined by equations (22) and (24) are also RF r-sep. sets as well from equation (25). That is, RF r-disconnectedness with respect to the fuzzy closure operator C_R implies RF r-disconnectedness and thus, RF r-connectedness implies RF r-connectedness with respect to the fuzzy closure operator C_R.

Example 3. Let $X = \{a, b, c, d\}$, $r = 0.6 \in I_0$, ℓ an F-ideal on X, and R a reflexive and transitive fuzzy relation defined by

R	a	b	c	d
a	1	0	0.4	0
b	0	1	0.4	0
c	0	0	1	0
d	0	0	0.4	1

$0.6 \vdash a \dashv = \{1, 0, 0.4, 0\}$, $0.6 \vdash b \dashv = \{0, 1, 0.4, 0\}$, $0.6 \vdash c \dashv = \{0, 0, 1, 0\}$, $0.6 \vdash d \dashv = \{0, 0, 0.4, 1\}$ and $\vdash a \dashv 0.6 = \{1, 0, 0, 0\}$, $\vdash b \dashv 0.6 = \{0, 1, 0, 0\}$, $\vdash c \dashv 0.6 = \{0.4, 0.4, 1, 0.4\}$, $\vdash d \dashv 0.6 = \{0, 0, 0, 1\}$. Then, $\vdash a \dashv R = \{1, 0, 0.4, 0\}$, $\vdash b \dashv R = \{0, 1, 0.4, 0\}$, $\vdash c \dashv R = \{0, 0, 0.4, 0\}$, $\vdash d \dashv R = \{0, 0, 0.4, 1\}$ and $R \vdash a \dashv = \{0.4, 0, 0, 0\}$, $R \vdash b \dashv = \{0, 0.4, 0, 0\}$, $R \vdash c \dashv = \{0.4, 0.4, 1, 0.4\}$, $R \vdash d \dashv = \{0, 0, 0, 0.4\}$. Then, $R \vdash a \dashv R = \{0.4, 0, 0, 0\}$, $R \vdash b \dashv R = \{0, 0.4, 0, 0\}$, $R \vdash c \dashv R = \{0, 0, 0.4, 0\}$, $R \vdash d \dashv R = \{0, 0, 0, 0.4\}$.

Consider $\ell(\nu) > 0 \forall \nu \leq \overline{0.5}$, otherwise $\ell(\nu) = 0$, then for any $\xi \in I^X$ we have $\ell(R \vdash x \dashv R \wedge \xi) > 0, \forall x \in X$, and thus $\xi^* = \overline{0}$. That is, $\xi^R = cl_R(\xi, r) = \xi$ for any $\xi \in I^X$. Hence, we could choose $\xi = \{0, 0, 0.5, 0.1\}$, $\zeta = \{0.7, 0.2, 0, 0\}$ so that the fuzzy set $(\xi \vee \zeta) = \{0.7, 0.2, 0.5, 0.1\}$ is a rough fuzzy set for which $cl_R(\xi, r) \wedge \zeta = \xi \wedge cl_R(\zeta, r) = \xi \wedge \zeta = \overline{0}$. Thus, {0.7, 0.2, 0.5, 0.1} is an RF r-disconnected set.

Note that: If we kept the above R and ℓ, r was any value in I_0, the above result will not be affected. But if we kept R and changed to an F-ideal ℓ' so that $\ell'(\nu) > 0, \forall \nu \leq \overline{0.3}$, otherwise $\ell'(\nu) = 0$, then the above result still not affected only if $r \leq 0.3$. Also if we kept the first *F-ideal* ℓ but we changed the three values "0.4" in the fuzzy relation R to be "0.5" then the above result still not affected only if $r \leq 0.5$.

The deduction is that the three choices of $r \in I_0, R, \ell$ play the main role in the computations of ξ^* for any $\xi \in I^X$ and then the resulting rough fuzzy set $\xi^R = cl_R(\xi, r)$, and so we could find a pair of RF r-sep. sets as shown above, and thus we can find a fuzzy set which is RF r-disconnected. Hence, in general, a fuzzy set is an RF r-connected set unless we restrict the choices of $r \in I_0, R, \ell$ in a particular way. Under this restriction, one can catch a pair of RF r-separated sets for which their union equal this fuzzy set.

Remark 18. Let (X, R) be an RFA space. Then, (X, R) is RF r-connected implies it is also RF r-connected with respect to the fuzzy closure operator C_R as defined in equations (24) and (26).

4. Conclusion

In this paper, we introduced the roughness of fuzzy sets with degree $r \in I_0$ for a fuzzy set $\xi \in I^X$ that explains the fuzzy roughness depending on the degree r of the fuzzy set ξ. Constructing rough sets in sense of Šostak is a good extension to the fuzzy rough sets defined before. Rough sets in this paper depend on three parameters. First is the ordinary relation or the fuzzy relation defined on X. Second is the ordinary ideal or the fuzzy ideal on X. Third is the degree $r \in [0, 1]$. This paper defined rough sets in sense of Šostak as a generalization of all definitions of rough sets given before. In the crisp case $r \in \{0, 1\}$, we are restricted to the definition of Kozae [7], which was a generalization of the definition of Kandil [4], which was a generalization of the previous basic definitions. From that point, this paper cures the gap between the roughness in sense of Chang and the roughness in sense of Šostak. This generalization concludes many of the previous definitions of rough fuzzy sets as special cases, and so it is as a good extension of the roughness in the fuzzy case.

Data Availability

The data sets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Alsulami and Saleh conducted the methodology and performed formal analysis; Abbas and Saleh validated the study and performed formal analysis; Abbas and Ibedou performed investigation for the study; Ibedou and Alsulami contributed to resources and conducted the methodology; Ibedou wrote the original draft; Ibedou and Abbas reviewed the final form. All the authors have agreed this published version of the manuscript.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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